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A COMPARISON OF THE EXACT AND APPROXIMATE POWER OF THE CHI-SQUARE GOODNESS-OF-FIT TEST

by

Brian Theodore Wright

United States
Naval Postgraduate School

THESIS

A COMPARISON OF THE EXACT AND APPROXIMATE POWER
OF THE CHI-SQUARE GOODNESS-OF-FIT TEST

by

Brian Theodore Wright

Thesis Advisor: R. R. Read

March 1971

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T137721

A Comparison of the Exact and Approximate Power of the Chi-square Goodness-of-fit Test

by

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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
March 1971

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ABSTRACT

This thesis presents a numerical comparison of the exact and approximate powers of the chi-square goodness-of-fit test for small numbers of classes and small sample sizes for the equiprobable null hypothesis. The comparison was performed using an IBM 360 computer and the computational details are presented within the thesis. In addition a comparison of critical points was conducted for the chi-square distribution and the associated exact, (multinomial), distribution. The results of the power comparisons show that the approximate power is surprisingly good and is recommended as an efficient method for determining type two error associated with the test. Further, use of the chi-square distribution for determining a critical point is reinforced through the numerical comparison of significance levels.

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I. INTRODUCTION

The power of the chi-square goodness-of-fit test has been an elusive problem which has attracted many authors. Eisenhart [1], Mann and Wald [2], and Patnaik [3] have all presented expressions for approximating the power of the test for simple null hypotheses. Later work by Mitra [4] and Diamond [5] presented power functions for compound null hypotheses. These approximate power functions have all been developed through theoretical considerations, however it is not known how good they are for approximating the true power of the chi-square test.

Cochran in his expository article [6] has presented a detailed history of the chi-square test. Included is a proposed method, (which he attributes to Tukey), for approximating the power of the chi-square test. This method has been referred to as the Pitman limiting power by a later author (cf. Mitra [4]). However the key idea appears to go back to Eisenhart.

Herein this approximation of power is compared with the true power as computed for the special case of the null hypothesis having equiprobable classes and alternative hypotheses such that all classes but one are equiprobable. It is shown that the approximation is reasonably good for small sample sizes.

It has long been recognized that the chi-square test provides only an approximate critical region. Thus comparisons of exact levels of significance with the approximate ones were also in order. Such a comparison of significance levels and their associated critical points is presented.

In the following section a discussion of the previous work performed in this area is presented along with notes on how this research fits into the scheme of the previous work. Section III presents the Eisenhart et al. approximation of the power and is followed in Section IV by the details of the special case used for the comparison of the exact and approximate power. The computational formulae for all the comparisons appear in Section V and the results and conclusions are discussed in Section VI.

II. DISCUSSION AND THE NATURE OF THE PROBLEM

In their 1931 paper Neyman and Pearson [7] presented an example of a three class multinomial probability function with a sample size of 10 observations. They observed that the probability calculations from the chi-square distribution were on a whole better than expected. However they opened to question the use of the chi-square approximation when the class expectations are small with respect to the sample size. This question has been answered only with hueristic suggestions in the literature, (cf. Cochran [8] and Watson [9]). The research reported within this paper sheds some additional light on this question.

Hoel in his 1938 paper [10] pointed out that there are two types of error associated with the chi-square goodness-of-fit test for small sample sizes. The first type of error arose from the fact that the derivation of the test criterion was based on rough approximations. Whereas the second type of error arose from using an integral of a continuous function instead of summing the appropriate terms of a discrete distribution to determine significance levels. Hoel concludes in his paper that errors based upon the derivation of the criterion from rough approximations are not significant. However he leaves untouched any discussion of how significant are the errors obtained by using an integral of a continuous function instead of summing the

discrete terms. Further, no research was uncovered which fully answered Hoel's question. This paper will help provide an answer to that question.

The majority of the work on the power function conducted in this field since 1945 has been concerned with theoretical developments using compound hypotheses. However Watson [9] in an expository article on recent results points out that the test has still not had any computations made regarding its power for small sample sizes, and he suggested some means of electronic calculation be performed to evaluate the power of the test. This in summary is what this paper presents.

III. APPROXIMATION TO POWER

The work reported in this section was proposed by several authors, however Eisenhart [1] is believed to be the first author to present the method. Hence the approximation to power is hereafter referred to as the approximation due to Eisenhart *et al.*

It is known that the chi-square goodness-of-fit test is a consistent test. As the number of observations taken from the sampled distribution increases, then the power of the test tends to unity for all alternative hypotheses. Thus the family of power curves might look like those in Figure 1 below, where ω_o represents the distribution specified by the null hypothesis.

Schematically one has $P\{\text{the test statistic} \geq \text{critical point}|\omega\} \to 1.0$ as $N \to \infty$ for each $\omega \varepsilon \{H_1\}$, the set of alternative hypotheses. In order to make this limit less than unity, it is necessary to choose a sequence of alternative hypotheses ω_N converging to ω_o as $N \to \infty$. The sequence $\beta_N(\omega_N) = P\{\text{the test statistic} \geq \text{critical point}|\omega_N\}$ might converge to some value $\beta < 1.0$ as indicated by the asterisks in Figure 1. By appropriate choice of the sequence ω_N the corresponding probabilities representing the power, $\beta_N(\omega_N)$, will converge to β rapidly (become and remain close to β for rather small N). Thus the power for finite N can be approximated by β. On the other hand, an inappropriately

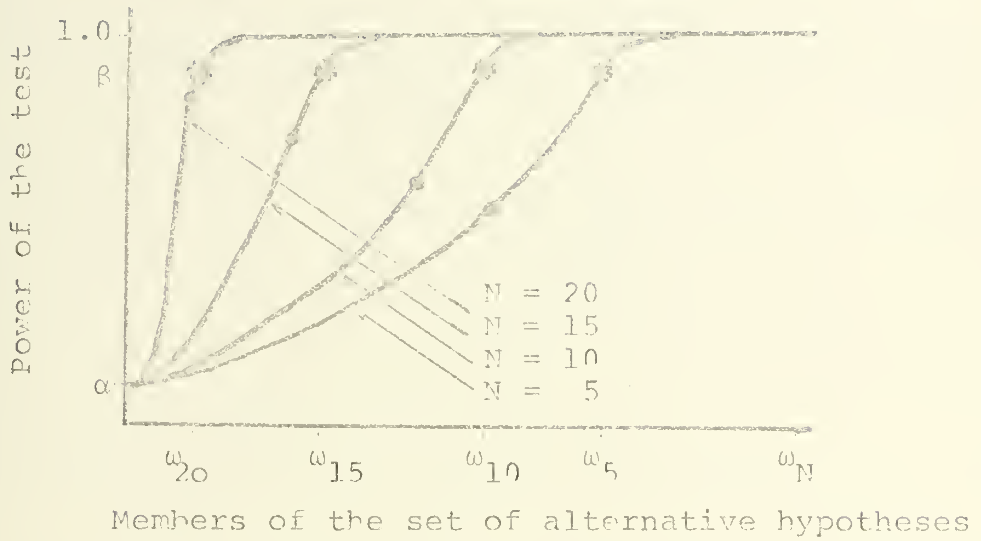

Figure 1. A Schematic Representation of Eisenhart, <u>et</u> <u>al</u>.'s Method of Approximating the Power.

chosen sequence ω_N' would not converge rapidly to β and hence β would be a poor approximation to $\beta_N(\omega_N')$ for many finite values of N. Such a sequence is indicated with dots in Figure 1. Thus the choice of sequences ω_N is critical.

The following expressions are presented for approximating the power (i.e., choosing the sequence) of the simple chi-square goodness-of-fit test.

Let the null hypothesis H_o describe k class probabilities $p_1,\ldots,p_k$, and let the alternative hypotheses $\{H_1\}$ be described by different choices of class probabilities, e.g., all possible $p_1^o,\ldots,p_k^o$ different from H_o. Define a term θ_i $i = 1,\ldots,k$ by $p_i^o = p_i + \theta_i/\sqrt{N}$ where N is the number of observations. Thus for a fixed alternative $p_1^o,\ldots,p_k^o$ and fixed N, the null hypothesis and the

alternative are connected by the $\{\theta_i\}$. As $N \to \infty$ then the p_i^o serve as the sequence ω_N and converge to p_i which serve as ω_o.

It is noted that $\sum_{i=1}^{k} p_i = 1 = \sum_{i=1}^{k} p_i^o$ hence $\sum_{i=1}^{k} \theta_i = 0$ and $\theta_i = \sqrt{N}\,(p_i^o - p_i)$.

Let x_i $i = 1,\ldots,k$ describe the observed frequency with which observations fall into frequency class i, then define a new term q_i as the observed portion of observations falling into class i, i.e. $q_i = x_i/N$.

The test statistic can therefore be defined to be

$$x^2 = \sum_{i=1}^{k} \left\{ \sqrt{N}\, \frac{(q_i - p_i)}{\sqrt{p_i}} \right\}^2 = \sum_{i=1}^{k} \left\{ \sqrt{N}\, \frac{(q_i - p_i^o)}{\sqrt{p_i^o}} \sqrt{\frac{p_i^o}{p_i}} + \frac{\theta_i}{\sqrt{p_i}} \right\}^2$$

$$= \sum_{i=1}^{k} \left\{ \sqrt{N}\, \frac{(q_i - p_i^o)}{\sqrt{p_i^o}} \sqrt{\frac{p_i^o}{p_i}} \right\}^2 + \sum_{i=1}^{k} \frac{\theta_i^2}{p_i}$$

with all cross product terms reduced to zero due to the restriction that $\sum_{i=1}^{k} \theta_i = 0$.

It was noted by Cochran [6], that the test statistic then has a non-central chi-square distribution (in the limit as $N \to \infty$) with non-centrality parameter

$$\lambda = \sum_{i=1}^{k} \frac{\theta_i^2}{p_i} .$$

IV. DETAILS OF THE SPECIAL CASE

As a sequence of alternative hypotheses had been proposed which converge to the null hypothesis, the following special case was developed to compare the approximation with the exact power of the chi-square test.

As Mann and Wald [2] pointed out in their paper, every continuous probability distribution can be transformed into a uniform distribution on the interval (0,1). Therefore the null hypothesis for the special case was that the classes of the multinomial were chosen such that they were described by equal class probabilities, i.e. $p_i = 1/k$, $i = 1,\ldots,k$ where k is the number of classes.

The only alternative hypotheses considered were those that specify equal probabilities for all classes but one. Since $\sum_{i=1}^{k} p_i^o = 1$, these alternative hypotheses may be represented as follows. Let $p_i^o = \rho/k$, $i = 1,\ldots,k-1$ and $p_k^o = 1 - (k-1)\rho/k$. Then the non-centrality parameter was computed from

$$\lambda = \sum_{i=1}^{k} \frac{\theta_i^2}{p_i} = N(k-1)(1-\rho)^2 .$$

The values of ρ used herein were .2, .5, .8.

The sequence of alternative hypotheses used for the comparison of the approximation was chosen due to its simplicity and rather extreme character; all but one cell being

equiprobable and the one cell having a surplus of probability. The same scheme with $\rho > 1$ would be less extreme. Obviously, the same value for the non-centrality parameter can be realized with other alternative hypotheses. It is proposed to research the question of whether or not the currently used scheme has a sense of extremity in terms of power when the non-centrality parameter is held fixed.

V. COMPUTATIONAL DETAILS

The problem was to compute the probability that the sum of squares of the class frequencies exceeded a predefined critical point. The work done previously in this area was mainly performed during the 1930's. At that time the size of the task was enormous and almost impossible since the researchers did not have the aid of modern electronic computation equipment.

It was decided that in order to gain enough information to have a meaningful presentation some type of computer program was required. The task involved writing a program which would generate the class partitions of the multinomial distribution and then compute the sum of squares for each of the generated partitions. Once the sum of squares had been computed the sum was checked to ensure that it was greater than the critical point as specified from the chi-square distribution. If the sum of squares did exceed the critical point then the multinomial probability associated with the partition was computed. If the sum was less than or equal to the critical point that partition was ignored and the program generated another partition and the process was repeated.

The multinomial probabilities associated with a fixed k part partition of N must be summed over all permutations of that partition. Thus

$$\sum_{i=1}^{k} \frac{N!}{x_1! \ldots x_k!} p_i^{o\,(N-x_i)} p_k^{o\,x_i} K_i$$

where K_i was the number of ways $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k$ could occur in the (k-1) remaining classes. The computations were performed in this manner due to the nature of the alternative hypotheses, with the first (k-1) classes of equal probability p_i^o and the k^{th} class of probability $p_k^o = (1 - (k-1)p_i^o)$.

One of the inefficiencies of the program was that it must compute K_i, the occurrence coefficient, k times for each of the generated partitions. This calculation required the computer to search through each generated partition and compute the number of occurrences of each possible x_i within the partition, and then determine the appropriate combinatoric.

To give an example of the foregoing, consider the following partition for a 6 class multinomial distribution with twelve observations, (5,3,3,1,0,0). The multinomial coefficient is $\frac{12!}{5!3!3!1!} = 665{,}280$. The partition probability under the alternative hypothesis is therefore $665{,}280\ (p_i^{o\,7} p_k^{o\,5} K_1 + p_i^{o\,9} p_k^{o\,3} K_2 + p_i^{o\,11} p_k^o K_3 + p_i^{o\,12} K_4)$ where K_i $i = 1,2,3,4$ are the associated occurrence coefficients for the x_i's. For the partition under consideration the K's were $K_1 = K_3 = \frac{5!}{2!1!2!} = 30$ $K_2 = K_4 = \frac{5!}{1!1!2!1!} = 60$. As a check it is noted that the total number of ways the partition could occur in a six class law is $\frac{6!}{1!2!2!1!} = 180$ and that $\sum_{i=1}^{4} K_i = 180$.

Having computed the probability under the alternative hypothesis that the test statistic is greater than the chi-square critical point, the non-centrality parameter lambda was computed for the k classes, the N observations and the appropriate ρ. With this parameter and (k-1) degrees of freedom the approximate power was computed utilizing the non-central chi-square distribution.

The computation of the approximate power was performed using a method found in Fix [11]. The power function of the chi-square goodness-of-fit test being approximated by

$$\beta(\lambda) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!} \int_{x^2_{(K-1)(\alpha)}}^{\infty} \frac{x^{k+2j-2}}{2^{\frac{1}{2}(k+2j-3)}\,\Gamma(\frac{k+1}{2}+j)} e^{-\frac{1}{2}x^2}\,dx,$$

where (k+2j-2) are the degrees of freedom for the incomplete gamma function and $x^2_{(k-1)(\alpha)}$ is the critical point of the chi-square with (k-1) degrees of freedom and the specified alpha level. The evaluation of the incomplete gamma function was accomplished by using a previously prepared program by John R. B. Whittlesey of UCLA, a listing of his program appears in the computer listing section of this paper.

Aside from the method employed to compute these probabilities there was nothing new in the theory employed. In fact this theory has been and continues to be the standard method of evaluating the power. The method used to generate the class partitions was original and was the important step in the process of allowing large amounts of data to be collected in a relatively small amount of time. The ordering

of the sum of squares of the k-part partitions of N is an important integer programming problem. Problems of this type are discussed in a survey article by Saaty [12].

Since the sum of squares of the generated partitions yields an integer value, the following method was used to calculate the critical point for the chi-square, and the exact critical points as determined under the equiprobable null hypothesis.

Let C_α be the critical point read from the chi-square table for (k-1) degrees of freedom (this is the value corresponding to a k class multinomial distribution). The probability that the test statistic exceeds this critical point is alpha, hence the following is true. If the test statistic is

$$\sum_{i=1}^{k} \frac{(x_i - Npi)^2}{Npi}$$

and

$$P \sum_{i=1}^{k} \left\{ \frac{(x_i - Npi)^2}{Npi} \geq C_\alpha \right\} = \alpha$$

as specified by the test, then

$$P \left\{ \sum_{i=1}^{k} x_i^2 \geq (C_\alpha + N) \frac{N}{k} \right\} = \alpha$$

since $p_i = 1/k$. Then since the sum of the squares of integers is again an integer the critical point of interest is the greatest integer in $[(C_\alpha + N)\ N/k]$.

The exact critical points were determined from the equiprobable multinomial by considering in turn each value for the sum of squares in decreasing order. Until a value was reached which yielded a probability slightly greater than the alpha probability. Then the process was repeated with the next smallest value such that the alpha level was bracketed. These two values became the upper critical point $\overline{CR}$ and the lower critical point $\underline{CR}$ respectively.

VI. RESULTS AND CONCLUSIONS

The results of the comparison of the approximate and exact powers of the chi-square test are found in Tables 2 through 17 in the computer output section of this thesis. These tables present the data in the following manner. For each value of ρ and alpha considered there are five columns; the first shows the number of classes k, the second the number of observations N, the third the exact power as computed from the associated k class multinomial distribution, the fourth displays the approximate power as computed by a method found in Fix [11], and the fifth column shows the associated non-centrality parameter for the approximation.

In order to provide a more concise display of information, four graphs, Figures 3 through 6, have been prepared which correspond to the data found in Tables 2 through 5. Several conclusions were drawn regarding the data found in the tables and the four graphs.

First it was noted that as the deviation, $(1 - \rho)$, increased between the null and alternative hypotheses the power of the test increased very rapidly with N. Secondly it was noted that the approximation of power was generally more conservative when the deviation between hypotheses was small, and that the approximation was generally over optimistic for large N and $(1 - \rho)$.

However it should be noted that as an approximation the asymptotic power is quite good especially as a means for determining how large a sample size is required to yield a specific level of power. Further, use of the non-central chi-square for estimating the probability of type II error associated with the test should be encouraged since it is an efficient method amenable to all alternative hypotheses.

The results of the comparison of significance levels of the exact critical points and their associated alpha levels are presented in Table 1. The data presented are for equiprobable multinomial distributions of three, four and five classes. Table 1 presents the data in the following manner. There are three divisions in the table, the first is a reference division showing class size and the number of observations, the second division is for the data associated with an alpha of .05, the third division is for the data associated with an alpha of .01. Within each of the latter two divisions there appear five columns; the first of these displays the greatest integer in the critical point calculation from the chi-square table (see section on computational details for further explanation), the second column presents the lower exact critical point, the third column presents the significance level associated with this critical point, in a like manner the fourth and fifth columns present the same data for the upper critical point.

Figure 2 presents a graphical representation of the data found in Table 1 for a four class multinomial distribution.

TABLE 1

CRITICAL POINTS COMPARISON FOR THE CENTRAL CHI-SQUARE AND THE EQUIPROBABLE MULTINOMIAL DISTRIBUTION

Class Size	# of Obs.	For Alpha 0.05: C_α from Chi-square	CR	$P(\Sigma x_i^2 \geq CR)$	$\overline{CR}$	$P(\Sigma x_i^2 \geq \overline{CR})$	For Alpha 0.01: C_α from Chi-square	CR	$P(\Sigma x_i^2 \geq CR)$	$\overline{CR}$	$P(\Sigma x_i^2 \geq \overline{CR})$
3	3	8	9	.11111	-	-	-	9	.11111	-	-
	4	13	10	.33333	16	.03704	-	16	.03704	-	-
	5	18	17	.13580	25	.01234	23	25	.01234	-	-
	6	23	26	.05349	36	.00411	30	26	.05349	36	.00411
	7	30	29	.07819	37	.02057	37	37	.02057	49	.00137
	8	37	38	.05898	40	.03337	45	40	.03337	50	.00777
	9	44	45	.05045	53	.01387	54	53	.01387	65	.00289
	10	53	54	.05899	58	.02241	64	66	.01021	68	.00564
	11	62	61	.05330	65	.03765	74	69	.02647	73	.00970
	12	71	72	.07044	74	.04845	84	86	.01157	90	.00411
	13	82	81	.06198	89	.03131	96	91	.01839	97	.00763
	14	93	94	.05822	98	.03310	108	108	.01211	110	.00835
	15	104	101	.09308	107	.04285	121	117	.01505	125	.00667
	16	117	118	.06149	120	.03916	134	134	.01324	136	.00715
	17	130	129	.06943	131	.04142	148	147	.01055	149	.00710
	18	143	146	.05655	150	.03253	163	164	.01005	170	.00506
	19	158	155	.06571	161	.04386	178	173	.01347	181	.00810
	20	173	174	.05555	176	.04515	194	194	.01347	198	.00707
4	4	11	10	.20313	16	.01563	15	16	.01563	-	-
	5	16	17	.06250	25	.00391	20	17	.06250	25	.00391
	6	20	20	.06250	26	.01855	25	26	.01855	36	.00097
	7	25	27	.05151	29	.02075	32	29	.02075	37	.00537
	8	31	30	.08868	34	.02075	38	38	.01691	40	.00665
	9	37	35	.09189	39	.04575	45	41	.03805	45	.00922
	10	44	44	.05151	46	.03710	53	54	.01128	58	.00303

TABLE 1 (Continued)

Class Size & # of Obs.		For Alpha 0.05					For Alpha 0.01				
		C_α from Chi-square	$\underline{CR}$	$P(\Sigma x_i^2 \geq \underline{CR})$	$\overline{CR}$	$P(\Sigma x_i^2 \geq \overline{CR})$	C_α from Chi-square	$\underline{CR}$	$P(\Sigma x_i^2 \geq \underline{CR})$	$\overline{CR}$	$P(\Sigma x_i^2 \geq \overline{CR})$
4	11	51	51	.05272	53	.04478	61	59	.01457	61	.00702
	12	59	58	.06527	60	.04827	69	74	.01042	80	.00192
	13	67	65	.06002	67	.04714	78	77	.01140	81	.00679
	14	76	76	.06010	78	.04588	88	90	.01044	92	.00614
	15	85	85	.05266	87	.03756	98	99	.01004	101	.00666
	16	95	94	.06079	96	.04335	109	108	.01034	110	.00899
	17	105	105	.05996	107	.03999	120	119	.01189	121	.00904
	18	116	116	.05070	118	.03894	131	130	.01199	132	.00908
	19	127	127	.05409	129	.04506	143	143	.01089	145	.00974
	20	139	138	.05852	140	.03982	156	154	.01307	156	.00868
5	5	14	13	.09760	17	.03360	18	17	.03360	25	.00160
	6	18	18	.09760	20	.02720	23	20	.02720	26	.00800
	7	23	21	.11296	25	.03232	28	27	.02336	29	.00723
	8	27	28	.05382	30	.03662	34	32	.01082	34	.00902
	9	33	33	.05339	35	.03210	40	41	.01136	45	.00243
	10	38	38	.05533	42	.03985	46	46	.01017	50	.00501
	11	45	45	.05032	47	.03159	53	53	.01267	55	.00920
	12	51	50	.07360	52	.03954	60	60	.01118	62	.00729
	13	58	57	.06350	59	.04535	68	67	.01169	69	.00832
	14	65	66	.05064	68	.03364	76	76	.01211	78	.00821
	15	73	71	.07437	73	.04674	84	83	.01239	85	.00903
	16	81	82	.05231	84	.03493	93	94	.01233	96	.00761
	17	90	89	.05623	91	.04675	103	103	.01077	105	.00722
	18	98	98	.06116	100	.04466	112	110	.01273	112	.00941
	19	108	109	.05015	111	.04196	122	123	.01099	125	.00957
	20	117	117	.05063	120	.03927	133	130	.01434	132	.00981

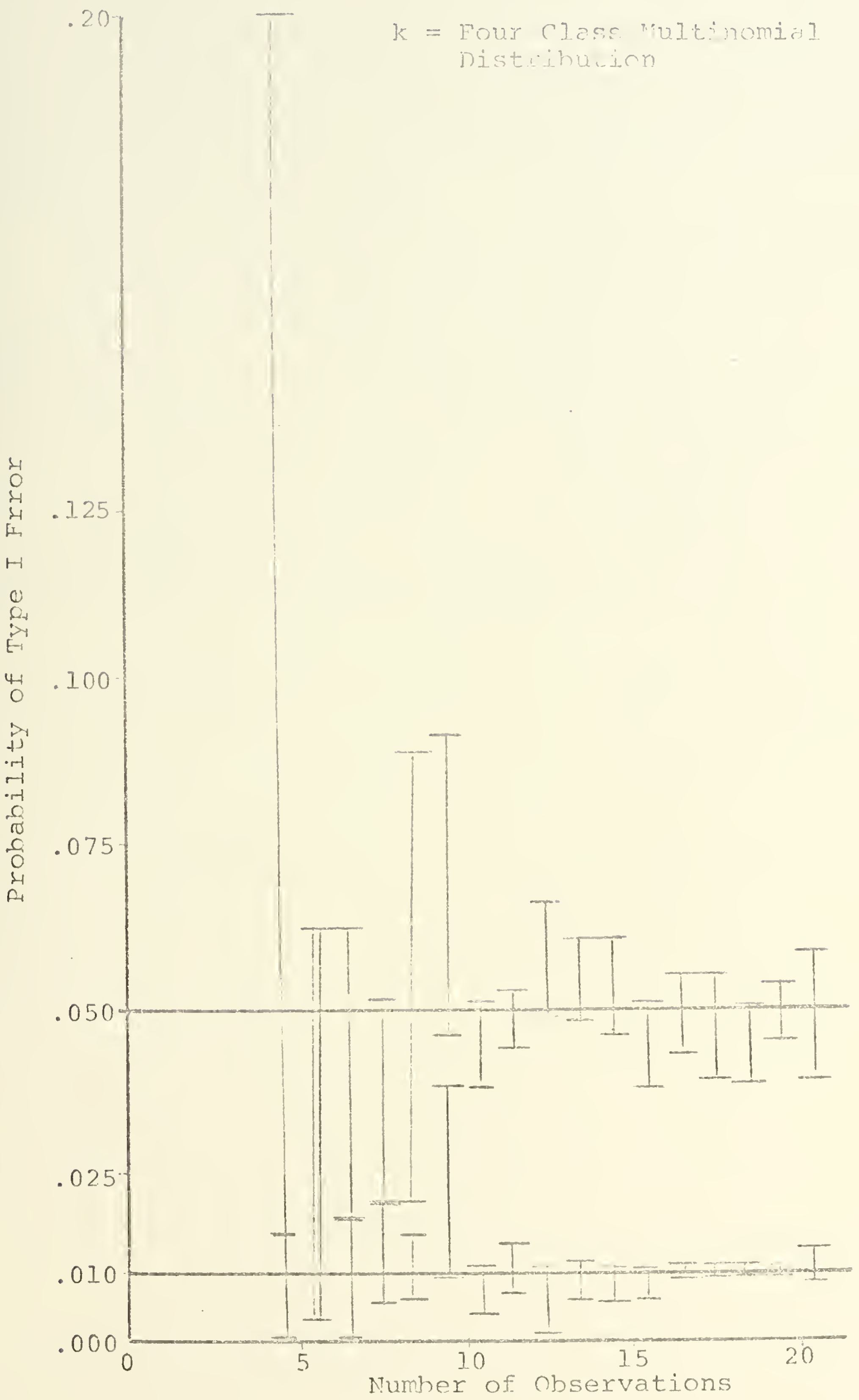

Figure 2. Significance Levels of Calculated Critical Points.

The upper and lower critical points straddle their respective alpha level and are connected by a straight line. In the case where the two alpha levels share common upper and lower critical points a double line connects the two associated significance levels.

The data presented in Table 1 and the graph, Figure 2, indicate that the critical point associated with the chi-square test is a very good approximation to the exact critical points, and is always bounded by the upper critical point. In those cases where it is not bounded by the lower critical point it does increase the probability of type I error, however this occurs infrequently.

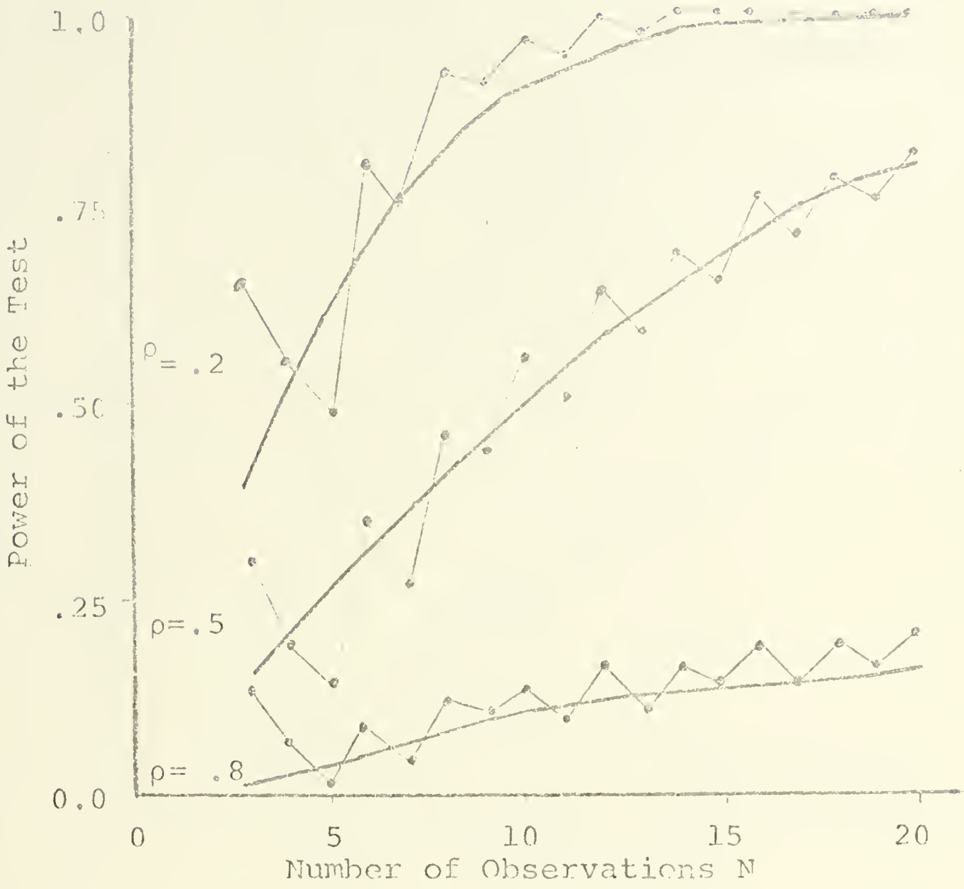

Figure 3. A Comparison of the Exact and Asymptotic Powers of the Chi-square Test for a Three Class Multinomial Distribution with an Alpha = .05.

The following remarks are applicable to Figures 3, 4, 5 and 6 only. The asymptotic or approximate power appears as the smooth curves in all figures, whereas the exact power curves are the jagged lines connecting the heavy dots. The ρ values indicated to the left of the curves were those associated with $p_i^o = \rho/k$ for the alternative hypotheses. The power of the test is plotted on the ordinate versus the number of observations, N, as plotted on the abcissa.

This figure corresponds to the data presented in Table 2.

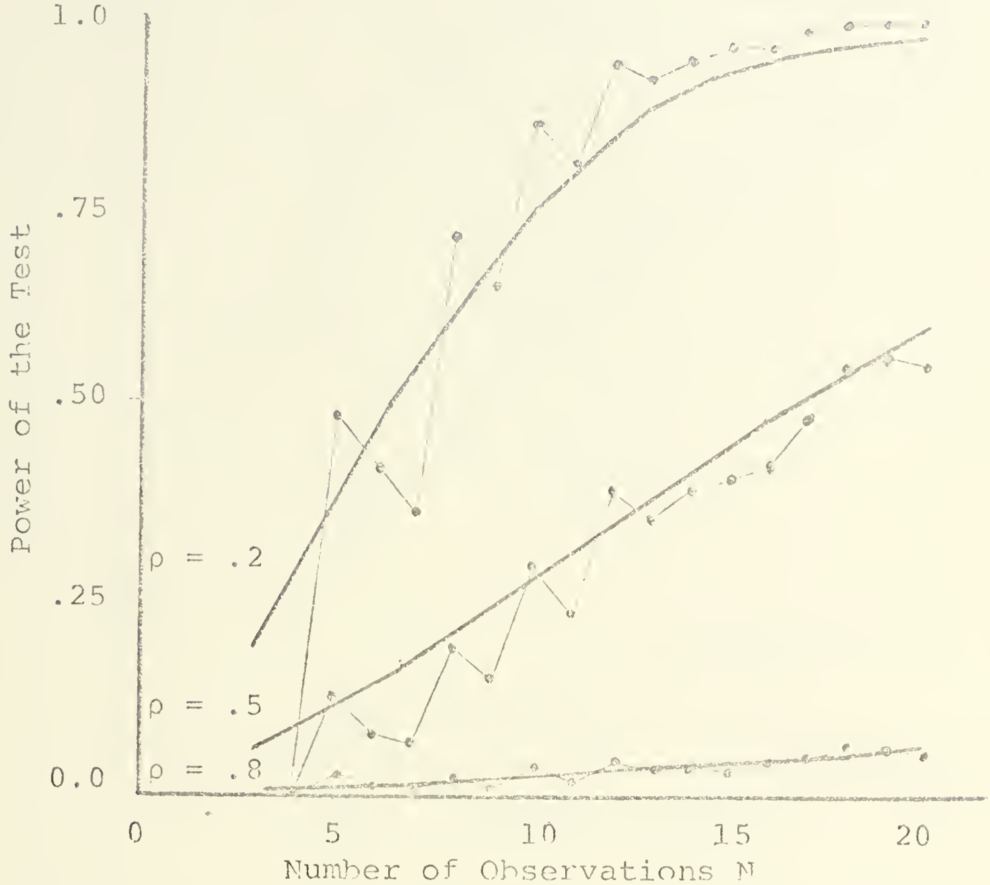

Figure 4. A Comparison of the Exact and Asymptotic Powers of the Chi-square Test for a Three Class Multinomial Distribution with an Alpha = .01.

See notes for Figure 3.

This figure corresponds to the data presented in Table 3.

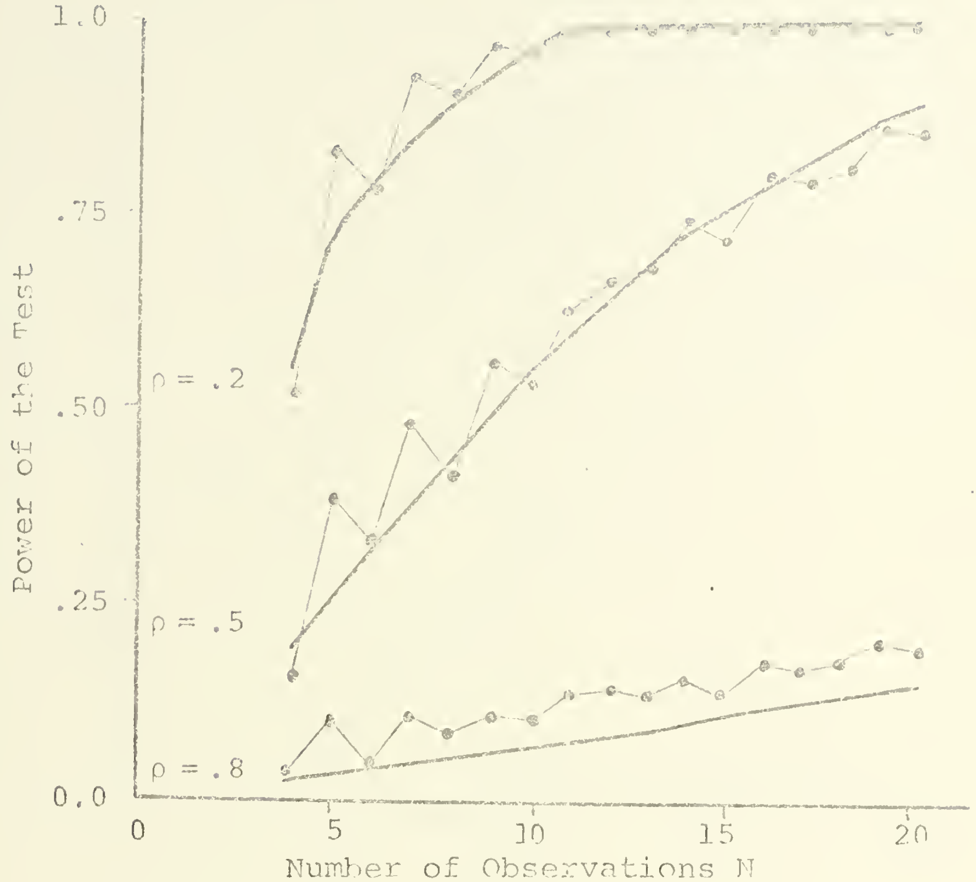

Figure 5. A Comparison of the Exact and Asymptotic Powers of the Chi-square Test for a Four Class Multinomial Distribution with an Alpha = .05.

See notes for Figure 3.

This figure corresponds to the data presented in Table 4.

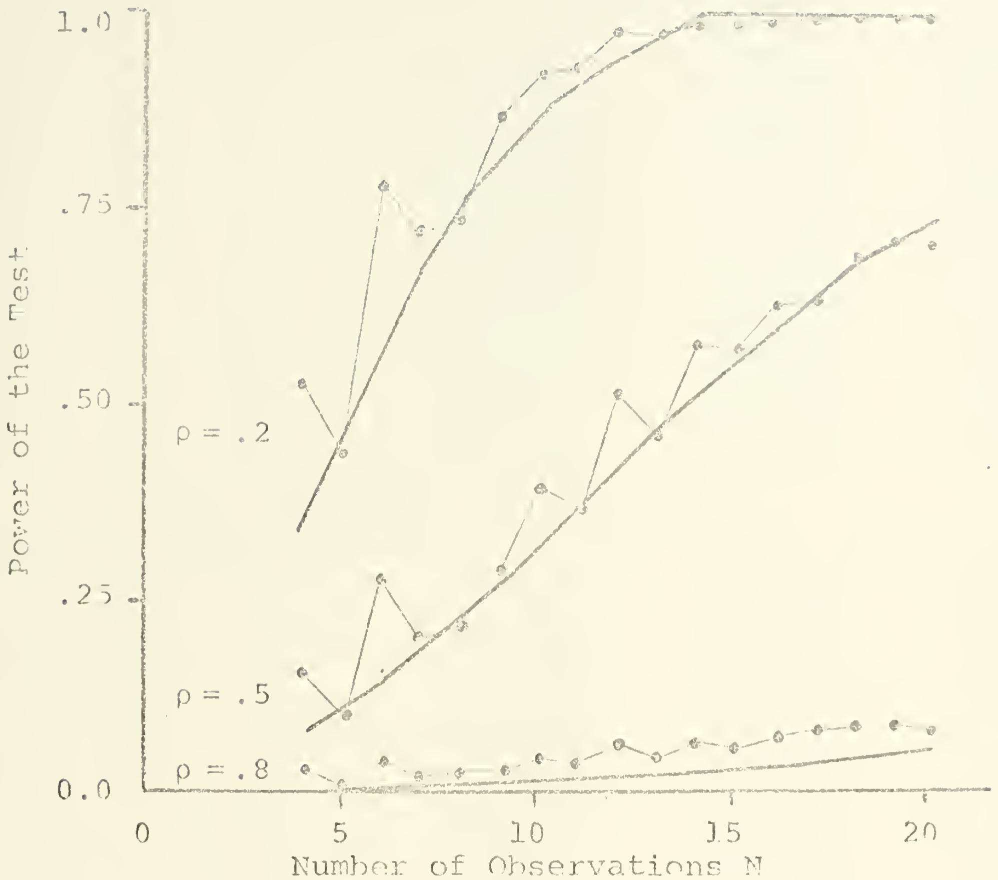

Figure 6. A Comparison of the Exact and Asymptotic Powers of the Chi-square Test for a Four Class Multinomial Distribution with an Alpha = .01.

See notes for Figure 3.

This figure corresponds to the data presented in Table 5.

TABLE 2. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.05 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
3	3	0.13956	0.02416	0.2400
3	4	0.05754	0.03223	0.3200
3	5	0.02483	0.04043	0.4000
3	6	0.09374	0.04862	0.4800
3	7	0.04726	0.05683	0.5600
3	8	0.11708	0.06507	0.6400
3	9	0.10546	0.07334	0.7200
3	10	0.13304	0.08163	0.8000
3	11	0.09900	0.08994	0.8800
3	12	0.16537	0.09826	0.9600
3	13	0.10993	0.10660	1.0400
3	14	0.16283	0.11496	1.1200
3	15	0.13477	0.12333	1.2000
3	16	0.18365	0.13171	1.2800
3	17	0.14074	0.14010	1.3600
3	18	0.19756	0.14849	1.4400
3	19	0.16205	0.15689	1.5200
3	20	0.20205	0.16529	1.6000

FOR ALPHA=.05 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
3	3	0.30556	0.15479	1.5000
3	4	0.19907	0.20723	2.0000
3	5	0.13194	0.25928	2.5000
3	6	0.35250	0.31047	3.0000
3	7	0.26363	0.36042	3.5000
3	8	0.46910	0.40878	4.0000
3	9	0.44673	0.45527	4.5000
3	10	0.55980	0.49968	5.0000
3	11	0.50295	0.54185	5.5000
3	12	0.64798	0.58167	6.0000
3	13	0.57061	0.61907	6.5000
3	14	0.69676	0.65404	7.0000
3	15	0.65994	0.68659	7.5000
3	16	0.75550	0.71675	8.0000
3	17	0.70068	0.74460	8.5000
3	18	0.78817	0.77022	9.0000
3	19	0.76348	0.79371	9.5000
3	20	0.82877	0.81517	10.0000

FOR ALPHA=.05 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
3	3	0.65156	0.39349	3.8400
3	4	0.56421	0.51001	5.1200
3	5	0.48895	0.61179	6.4000
3	6	0.81492	0.69772	7.6800
3	7	0.76276	0.76825	8.9600
3	8	0.92096	0.82478	10.2400
3	9	0.91393	0.86917	11.5200
3	10	0.96596	0.90342	12.8000
3	11	0.95682	0.92944	14.0800
3	12	0.98606	0.94893	15.3600
3	13	0.97978	0.96336	16.6400
3	14	0.99373	0.97392	17.9200
3	15	0.99216	0.98157	19.2000
3	16	0.99746	0.98707	20.4800
3	17	0.99621	0.99099	21.7600
3	18	0.99883	0.99375	23.0400
3	19	0.99855	0.99569	24.3200
3	20	0.99953	0.99705	25.6000

TABLE 3. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.01 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
3	3	0.0	0.00709	0.2400
3	4	0.0	0.00961	0.3200
3	5	0.02483	0.01222	0.4000
3	6	0.01105	0.01490	0.4800
3	7	0.00501	0.01767	0.5600
3	8	0.02399	0.02052	0.6400
3	9	0.01220	0.02345	0.7200
3	10	0.03622	0.02646	0.8000
3	11	0.01996	0.02954	0.8800
3	12	0.04712	0.03271	0.9600
3	13	0.03567	0.03596	1.0400
3	14	0.04144	0.03928	1.1200
3	15	0.03895	0.04269	1.2000
3	16	0.04069	0.04617	1.2800
3	17	0.04567	0.04972	1.3600
3	18	0.06079	0.05335	1.4400
3	19	0.05796	0.05706	1.5200
3	20	0.05416	0.06084	1.6000

FOR ALPHA=.01 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
3	3	0.0	0.05613	1.5000
3	4	0.0	0.08081	2.0000
3	5	0.13194	0.10812	2.5000
3	6	0.08783	0.13775	3.0000
3	7	0.05853	0.16936	3.5000
3	8	0.19514	0.20261	4.0000
3	9	0.14308	0.23713	4.5000
3	10	0.29918	0.27256	5.0000
3	11	0.23412	0.30857	5.5000
3	12	0.39310	0.34483	6.0000
3	13	0.35116	0.38106	6.5000
3	14	0.39515	0.41698	7.0000
3	15	0.41748	0.45235	7.5000
3	16	0.42304	0.48696	8.0000
3	17	0.48625	0.52064	8.5000
3	18	0.55078	0.55324	9.0000
3	19	0.56675	0.58464	9.5000
3	20	0.56355	0.61474	10.0000

FOR ALPHA=.01 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
3	3	0.0	0.19182	3.8400
3	4	0.0	0.28116	5.1200
3	5	0.48895	0.37383	6.4000
3	6	0.42375	0.46490	7.6800
3	7	0.36725	0.55068	8.9600
3	8	0.71002	0.62870	10.2400
3	9	0.65779	0.69764	11.5200
3	10	0.86149	0.75708	12.8000
3	11	0.82754	0.80723	14.0800
3	12	0.93543	0.84876	15.3600
3	13	0.92404	0.88259	16.6400
3	14	0.94191	0.90975	17.9200
3	15	0.96208	0.93125	19.2000
3	16	0.96291	0.94807	20.4800
3	17	0.98183	0.96109	21.7600
3	18	0.98804	0.97106	23.0400
3	19	0.99217	0.97863	24.3200
3	20	0.99226	0.98432	25.6000

TABLE 4. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.05 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
4	4	0.03040	0.02526	0.4800
4	5	0.10720	0.03205	0.6000
4	6	0.04576	0.03902	0.7200
4	7	0.11027	0.04617	0.8400
4	8	0.08070	0.05349	0.9600
4	9	0.11823	0.06098	1.0800
4	10	0.10391	0.06863	1.2000
4	11	0.13159	0.07644	1.3200
4	12	0.14386	0.08441	1.4400
4	13	0.13319	0.09252	1.5600
4	14	0.16198	0.10077	1.6800
4	15	0.14373	0.10915	1.8000
4	16	0.17528	0.11766	1.9200
4	17	0.17019	0.12630	2.0400
4	18	0.17643	0.13505	2.1600
4	19	0.20644	0.14391	2.2800
4	20	0.19734	0.15287	2.4000

FOR ALPHA=.05 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
4	4	0.15332	0.19900	3.0000
4	5	0.38477	0.25885	3.7500
4	6	0.27467	0.31975	4.5000
4	7	0.47689	0.38041	5.2500
4	8	0.41039	0.43970	6.0000
4	9	0.55701	0.49673	6.7500
4	10	0.53256	0.55083	7.5000
4	11	0.63100	0.60150	8.2500
4	12	0.65833	0.64844	9.0000
4	13	0.68135	0.69150	9.7500
4	14	0.74564	0.73063	10.5000
4	15	0.72829	0.76592	11.2500
4	16	0.80150	0.79749	12.0000
4	17	0.79943	0.82554	12.7500
4	18	0.82551	0.85032	13.5000
4	19	0.86713	0.87207	14.2500
4	20	0.86265	0.89105	15.0000

FOR ALPHA=.05 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
4	4	0.52203	0.56332	7.6800
4	5	0.83530	0.68320	9.6000
4	6	0.77649	0.77770	11.5200
4	7	0.92625	0.84845	13.4400
4	8	0.90482	0.89926	15.3600
4	9	0.96718	0.93451	17.2800
4	10	0.96351	0.95827	19.2000
4	11	0.98592	0.97387	21.1200
4	12	0.98844	0.98390	23.0400
4	13	0.99360	0.99023	24.9600
4	14	0.99679	0.99415	26.8800
4	15	0.99712	1.00000	28.8000
4	16	0.99907	1.00000	30.7200
4	17	0.99907	1.00000	32.6400
4	18	0.99956	1.00000	34.5600
4	19	0.99981	1.00000	36.4800
4	20	0.99983	1.00000	38.4000

TABLE 5. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.01 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
4	4	0.03040	0.00659	0.4800
4	5	0.01120	0.00856	0.6000
4	6	0.04576	0.01066	0.7200
4	7	0.01996	0.01289	0.8400
4	8	0.02383	0.01526	0.9600
4	9	0.02598	0.01777	1.0800
4	10	0.04748	0.02042	1.2000
4	11	0.03295	0.02320	1.3200
4	12	0.06137	0.02613	1.4400
4	13	0.04302	0.02920	1.5600
4	14	0.06531	0.03241	1.6800
4	15	0.06303	0.03577	1.8000
4	16	0.06785	0.03926	1.9200
4	17	0.07013	0.04290	2.0400
4	18	0.07457	0.04668	2.1600
4	19	0.08326	0.05060	2.2800
4	20	0.07278	0.05466	2.4000

FOR ALPHA=.01 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
4	4	0.15332	0.07699	3.0000
4	5	0.09546	0.10946	3.7500
4	6	0.27467	0.14644	4.5000
4	7	0.19379	0.18722	5.2500
4	8	0.21342	0.23102	6.0000
4	9	0.28177	0.27700	6.7500
4	10	0.39447	0.32435	7.5000
4	11	0.36744	0.37229	8.2500
4	12	0.51378	0.42013	9.0000
4	13	0.45595	0.46722	9.7500
4	14	0.57442	0.51304	10.5000
4	15	0.56431	0.55715	11.2500
4	16	0.62399	0.59919	12.0000
4	17	0.63416	0.63892	12.7500
4	18	0.67375	0.67614	13.5000
4	19	0.71001	0.71076	14.2500
4	20	0.70840	0.74272	15.0000

FOR ALPHA=.01 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
4	4	0.52203	0.33583	7.6800
4	5	0.44371	0.45789	9.6000
4	6	0.77649	0.57254	11.5200
4	7	0.71658	0.67326	13.4400
4	8	0.73639	0.75713	15.3600
4	9	0.85915	0.82396	17.2800
4	10	0.92118	0.87528	19.2000
4	11	0.93254	0.91344	21.1200
4	12	0.97633	0.94105	23.0400
4	13	0.96948	0.96054	24.9600
4	14	0.98897	1.00000	26.8800
4	15	0.98831	1.00000	28.8000
4	16	0.99477	1.00000	30.7200
4	17	0.99517	1.00000	32.6400
4	18	0.99760	1.00000	34.5600
4	19	0.99836	1.00000	36.4800
4	20	0.99885	1.00000	38.4000

TABLE 6. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.05 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
5	5	0.07122	0.05987	0.8000
5	6	0.06149	0.07205	0.9600
5	7	0.08421	0.08431	1.1200
5	8	0.13661	0.09664	1.2800
5	9	0.09850	0.10904	1.4400
5	10	0.13028	0.12152	1.6000
5	11	0.11505	0.13406	1.7600
5	12	0.14248	0.14666	1.9200
5	13	0.17443	0.15932	2.0800
5	14	0.18667	0.17203	2.2400
5	15	0.18034	0.18478	2.4000
5	16	0.22095	0.19758	2.5600
5	17	0.20918	0.21040	2.7200
5	18	0.21753	0.22324	2.8800
5	19	0.24533	0.23611	3.0400
5	20	0.25381	0.24898	3.2000

FOR ALPHA=.05 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
5	5	0.33880	0.39193	5.0000
5	6	0.31360	0.46745	6.0000
5	7	0.44182	0.53835	7.0000
5	8	0.59945	0.60365	8.0000
5	9	0.53582	0.66278	9.0000
5	10	0.64820	0.71551	10.0000
5	11	0.62775	0.76189	11.0000
5	12	0.70302	0.80219	12.0000
5	13	0.78344	0.83681	13.0000
5	14	0.78785	0.86625	14.0000
5	15	0.81603	0.89105	15.0000
5	16	0.86716	0.91177	16.0000
5	17	0.85973	0.92894	17.0000
5	18	0.88771	0.94307	18.0000
5	19	0.90873	0.95461	19.0000
5	20	0.91907	0.96398	20.0000

FOR ALPHA=.05 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
5	5	0.81656	0.83032	12.8000
5	6	0.80069	0.89895	15.3600
5	7	0.91806	0.94204	17.9200
5	8	0.97371	0.96781	20.4800
5	9	0.96504	0.98263	23.0400
5	10	0.98771	0.99086	25.6000
5	11	0.98636	0.99530	28.1600
5	12	0.99458	1.00000	30.7200
5	13	0.99825	1.00000	33.2800
5	14	0.99827	1.00000	35.8400
5	15	0.99923	1.00000	38.4000
5	16	0.99972	1.00000	40.9600
5	17	0.99972	1.00000	43.5200
5	18	0.99989	1.00000	46.0800
5	19	0.99994	1.00000	48.6400
5	20	0.99996	1.00000	51.2000

TABLE 7. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.01 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
5	5	0.00647	0.01756	0.8000
5	6	0.02758	0.02162	0.9600
5	7	0.02576	0.02588	1.1200
5	8	0.03066	0.03034	1.2800
5	9	0.04878	0.03500	1.4400
5	10	0.03348	0.03986	1.6000
5	11	0.05652	0.04492	1.7600
5	12	0.04749	0.05019	1.9200
5	13	0.06267	0.05566	2.0800
5	14	0.06439	0.06134	2.2400
5	15	0.07398	0.06721	2.4000
5	16	0.09283	0.07329	2.5600
5	17	0.07398	0.07956	2.7200
5	18	0.09030	0.08603	2.8800
5	19	0.11343	0.09268	3.0400
5	20	0.10792	0.09953	3.2000

FOR ALPHA=.01 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
5	5	0.07780	0.18798	5.0000
5	6	0.23350	0.24425	6.0000
5	7	0.22430	0.30363	7.0000
5	8	0.31549	0.36450	8.0000
5	9	0.39334	0.42540	9.0000
5	10	0.38715	0.48505	10.0000
5	11	0.51170	0.54239	11.0000
5	12	0.48289	0.59659	12.0000
5	13	0.58757	0.64709	13.0000
5	14	0.59914	0.69349	14.0000
5	15	0.65311	0.73563	15.0000
5	16	0.73049	0.77347	16.0000
5	17	0.69224	0.80710	17.0000
5	18	0.75843	0.83671	18.0000
5	19	0.80862	0.86256	19.0000
5	20	0.80954	0.88495	20.0000

FOR ALPHA=.01 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
5	5	0.41821	0.63731	12.8000
5	6	0.75278	0.74974	15.3600
5	7	0.74476	0.83448	17.9200
5	8	0.87740	0.89455	20.4800
5	9	0.91322	0.93501	23.0400
5	10	0.93944	0.96112	25.6000
5	11	0.97334	1.00000	28.1600
5	12	0.97355	1.00000	30.7200
5	13	0.99017	1.00000	33.2800
5	14	0.99097	1.00000	35.8400
5	15	0.99577	1.00000	38.4000
5	16	0.99850	1.00000	40.9600
5	17	0.99800	1.00000	43.5200
5	18	0.99928	1.00000	46.0800
5	19	0.99965	1.00000	48.6400
5	20	0.99973	1.00000	51.2000

TABLE 8. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.05 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
6	6	0.12897	0.05620	1.2000
6	7	0.05646	0.06658	1.4000
6	8	0.13134	0.07725	1.6000
6	9	0.14660	0.08821	1.8000
6	10	0.13494	0.09945	2.0000
6	11	0.15497	0.11095	2.2000
6	12	0.14266	0.12272	2.4000
6	13	0.16648	0.13472	2.6000
6	14	0.18575	0.14696	2.8000
6	15	0.19847	0.15942	3.0000
6	16	0.19492	0.17209	3.2000
6	17	0.22487	0.18494	3.4000
6	18	0.25010	0.19797	3.6000
6	19	0.23648	0.21115	3.8000
6	20	0.27689	0.22448	4.0000

FOR ALPHA=.05 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
6	6	0.52450	0.46396	7.5000
6	7	0.39794	0.54410	8.7500
6	8	0.59389	0.61767	10.0000
6	9	0.65611	0.68353	11.2500
6	10	0.65761	0.74119	12.5000
6	11	0.73460	0.79071	13.7500
6	12	0.72763	0.83251	15.0000
6	13	0.78069	0.86726	16.2500
6	14	0.83322	0.89576	17.5000
6	15	0.84412	0.91883	18.7500
6	16	0.86201	0.93730	20.0000
6	17	0.89879	0.95192	21.2500
6	18	0.91686	0.96340	22.5000
6	19	0.91846	0.97232	23.7500
6	20	0.94311	0.97920	25.0000

FOR ALPHA=.05 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
6	6	0.93995	0.92597	19.2000
6	7	0.90748	0.96258	22.4000
6	8	0.97303	0.98191	25.6000
6	9	0.98382	1.00000	28.8000
6	10	0.98837	1.00000	32.0000
6	11	0.99580	1.00000	35.2000
6	12	0.99569	1.00000	38.4000
6	13	0.99824	1.00000	41.6000
6	14	0.99939	1.00000	44.8000
6	15	0.99947	1.00000	48.0000
6	16	0.99975	1.00000	51.2000
6	17	0.99992	1.00000	54.4000
6	18	0.99995	1.00000	57.6000
6	19	0.99997	1.00000	60.8000
6	20	0.99999	1.00000	64.0000

TABLE 9. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.01 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
6	6	0.01896	0.01546	1.2000
6	7	0.04877	0.01836	1.4000
6	8	0.02499	0.02253	1.6000
6	9	0.04524	0.02645	1.8000
6	10	0.07537	0.03065	2.0000
6	11	0.04993	0.03512	2.2000
6	12	0.06499	0.03988	2.4000
6	13	0.07995	0.04491	2.6000
6	14	0.08234	0.05022	2.8000
6	15	0.08953	0.05582	3.0000
6	16	0.08850	0.06170	3.2000
6	17	0.10750	0.06786	3.4000
6	18	0.11988	0.07429	3.6000
6	19	0.11626	0.08100	3.8000
6	20	0.12507	0.08799	4.0000

FOR ALPHA=.01 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
6	6	0.20837	0.24699	7.5000
6	7	0.38452	0.31517	8.7500
6	8	0.29472	0.38544	10.0000
6	9	0.44362	0.45553	11.2500
6	10	0.55132	0.52348	12.5000
6	11	0.51073	0.58777	13.7500
6	12	0.61700	0.64730	15.0000
6	13	0.64164	0.70136	16.2500
6	14	0.68468	0.74962	17.5000
6	15	0.72362	0.79204	18.7500
6	16	0.73490	0.82880	20.0000
6	17	0.79040	0.86024	21.2500
6	18	0.83205	0.88681	22.5000
6	19	0.83028	0.90902	23.7500
6	20	0.86053	1.00000	25.0000

FOR ALPHA=.01 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
6	6	0.73678	0.80591	19.2000
6	7	0.90423	0.88485	22.4000
6	8	0.86954	1.00000	25.6000
6	9	0.95230	1.00000	28.8000
6	10	0.97457	1.00000	32.0000
6	11	0.97747	1.00000	35.2000
6	12	0.99117	1.00000	38.4000
6	13	0.99294	1.00000	41.6000
6	14	0.99665	1.00000	44.8000
6	15	0.99794	1.00000	48.0000
6	16	0.99859	1.00000	51.2000
6	17	0.99947	1.00000	54.4000
6	18	0.99977	1.00000	57.6000
6	19	0.99979	1.00000	60.8000
6	20	0.99992	1.00000	64.0000

TABLE 10. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.05 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
7	7	0.09901	0.10669	1.6800
7	8	0.11610	0.12232	1.9200
7	9	0.14223	0.13807	2.1600
7	10	0.13490	0.15396	2.4000
7	11	0.13981	0.16998	2.6400
7	12	0.17920	0.18611	2.8800
7	13	0.18333	0.20235	3.1200
7	14	0.20577	0.21869	3.3600
7	15	0.20608	0.23511	3.6000
7	16	0.25202	0.25159	3.8400
7	17	0.24901	0.26811	4.0800
7	18	0.26159	0.28467	4.3200
7	19	0.29732	0.30124	4.5600
7	20	0.29753	0.31780	4.8000

FOR ALPHA=.05 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
7	7	0.49341	0.66851	10.5000
7	8	0.57324	0.73747	12.0000
7	9	0.68449	0.79527	13.5000
7	10	0.66817	0.84260	15.0000
7	11	0.72628	0.88057	16.5000
7	12	0.80300	0.91048	18.0000
7	13	0.81511	0.93365	19.5000
7	14	0.84956	0.95134	21.0000
7	15	0.87770	0.96467	22.5000
7	16	0.91035	0.97458	24.0000
7	17	0.91481	0.98186	25.5000
7	18	0.93140	0.98717	27.0000
7	19	0.94935	0.99099	28.5000
7	20	0.95508	1.00000	30.0000

FOR ALPHA=.05 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
7	7	0.93447	0.98680	26.8800
7	8	0.97112	1.00000	30.7200
7	9	0.99042	1.00000	34.5600
7	10	0.98922	1.00000	38.3999
7	11	0.99566	1.00000	42.2399
7	12	0.99853	1.00000	46.0800
7	13	0.99882	1.00000	49.9199
7	14	0.99949	1.00000	53.7599
7	15	0.99980	1.00000	57.5999
7	16	0.99991	1.00000	61.4399
7	17	0.99994	1.00000	65.2799
7	18	0.99998	1.00000	69.1199
7	19	0.99999	1.00000	72.9599
7	20	0.99999	1.00000	76.7999

TABLE 11. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.01 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
7	7	0.04166	0.03355	1.6800
7	8	0.04244	0.03958	1.9200
7	9	0.042[illegible]2	0.04596	2.1600
7	10	0.06382	0.05269	2.4000
7	11	0.06263	0.05978	2.6400
7	12	0.07466	0.06722	2.8800
7	13	0.08791	0.07501	3.1200
7	14	0.10225	0.08316	3.3600
7	15	0.09615	0.09165	3.6000
7	16	0.11426	0.10049	3.8400
7	17	0.12698	0.10966	4.0800
7	18	0.12864	0.11916	4.3200
7	19	0.13622	0.12897	4.5600
7	20	0.15174	0.13909	4.8000

FOR ALPHA=.01 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
7	7	0.36874	0.43344	10.5000
7	8	0.37956	0.51365	12.0000
7	9	0.43787	0.58916	13.5000
7	10	0.56521	0.65816	15.0000
7	11	0.56054	0.71958	16.5000
7	12	0.63207	0.77302	18.0000
7	13	0.70813	0.81857	19.5000
7	14	0.73936	0.85668	21.0000
7	15	0.75457	0.88803	22.5000
7	16	0.81195	0.91344	24.0000
7	17	0.83415	0.93374	25.5000
7	18	0.84937	1.00000	27.0000
7	19	0.87789	1.00000	28.5000
7	20	0.89761	1.00000	30.0000

FOR ALPHA=.01 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
7	7	0.89980	1.00000	26.8800
7	8	0.90520	1.00000	30.7200
7	9	0.95153	1.00000	34.5600
7	10	0.98290	1.00000	38.3999
7	11	0.98244	1.00000	42.2399
7	12	0.99251	1.00000	46.0800
7	13	0.99713	1.00000	49.9199
7	14	0.99790	1.00000	53.7599
7	15	0.99882	1.00000	57.5999
7	16	0.99959	1.00000	61.4399
7	17	0.99971	1.00000	65.2799
7	18	0.99984	1.00000	69.1199
7	19	0.99994	1.00000	72.9599
7	20	0.99996	1.00000	76.7999

TABLE 12. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.05 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
8	8	0.08422	0.09899	2.2400
8	9	0.14598	0.11311	2.5200
8	10	0.15325	0.12763	2.8000
8	11	0.19996	0.14253	3.0800
8	12	0.19065	0.15780	3.3600
8	13	0.21574	0.17240	3.6400
8	14	0.22146	0.18932	3.9200
8	15	0.25122	0.20553	4.2000
8	16	0.23946	0.22200	4.4800
8	17	0.27699	0.23870	4.7600
8	18	0.27118	0.25560	5.0400
8	19	0.30738	0.27266	5.3200
8	20	0.31637	0.28986	5.6000

FOR ALPHA=.05 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
8	8	0.54111	0.74563	14.0000
8	9	0.68807	0.80732	15.7500
8	10	0.73683	0.85666	17.5000
8	11	0.79125	0.89512	19.2500
8	12	0.81561	0.92442	21.0000
8	13	0.86263	0.94630	22.7500
8	14	0.88081	0.96235	24.5000
8	15	0.90123	0.97392	26.2500
8	16	0.91431	1.00000	28.0000
8	17	0.93827	1.00000	29.7500
8	18	0.93937	1.00000	31.5000
8	19	0.95649	1.00000	33.2500
8	20	0.96613	1.00000	35.0000

FOR ALPHA=.05 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
8	8	0.96773	1.00000	35.8400
8	9	0.99055	1.00000	40.3200
8	10	0.99458	1.00000	44.8000
8	11	0.99737	1.00000	49.2800
8	12	0.99877	1.00000	53.7600
8	13	0.99957	1.00000	58.2400
8	14	0.99971	1.00000	62.7200
8	15	0.99986	1.00000	67.2000
8	16	0.99994	1.00000	71.6800
8	17	0.99997	1.00000	76.1599
8	18	0.99998	1.00000	80.6400
8	19	0.99999	1.00000	85.1199
8	20	1.00000	1.00000	89.6000

TABLE 13. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.01 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
8	8	0.06140	0.02972	2.2400
8	9	0.07995	0.03512	2.5200
8	10	0.06917	0.04096	2.8000
8	11	0.08748	0.04723	3.0800
8	12	0.09934	0.05395	3.3600
8	13	0.07922	0.06110	3.6400
8	14	0.11117	0.06870	3.9200
8	15	0.12913	0.07675	4.2000
8	16	0.11776	0.08523	4.4800
8	17	0.12946	0.09415	4.7600
8	18	0.14399	0.10350	5.0400
8	19	0.14432	0.11326	5.3200
8	20	0.15537	0.12342	5.6000

FOR ALPHA=.01 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
8	8	0.50730	0.52733	14.0000
8	9	0.56285	0.60934	15.7500
8	10	0.57558	0.68314	17.5000
8	11	0.67278	0.74750	19.2500
8	12	0.69825	0.80209	21.0000
8	13	0.69874	0.84727	22.7500
8	14	0.78465	0.88383	24.5000
8	15	0.81812	1.00000	26.2500
8	16	0.81844	1.00000	28.0000
8	17	0.85617	1.00000	29.7500
8	18	0.87912	1.00000	31.5000
8	19	0.88868	1.00000	33.2500
8	20	0.91063	1.00000	35.0000

FOR ALPHA=.01 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
8	8	0.96394	1.00000	35.8400
8	9	0.97326	1.00000	40.3200
8	10	0.98363	1.00000	44.8000
8	11	0.99427	1.00000	49.2800
8	12	0.99526	1.00000	53.7600
8	13	0.99711	1.00000	58.2400
8	14	0.99910	1.00000	62.7200
8	15	0.99943	1.00000	67.2000
8	16	0.99962	1.00000	71.6800
8	17	0.99986	1.00000	76.1599
8	18	0.99991	1.00000	80.6400
8	19	0.99995	1.00000	85.1199
8	20	0.99998	1.00000	89.6000

TABLE 14. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.05 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
9	9	0.14996	0.16394	2.8800
9	10	0.16610	0.18290	3.2000
9	11	0.20937	0.20207	3.5200
9	12	0.19379	0.22142	3.8400
9	13	0.19679	0.24093	4.1600
9	14	0.26136	0.26057	4.4800
9	15	0.28415	0.28032	4.8000
9	16	0.29980	0.30014	5.1200
9	17	0.27318	0.31999	5.4400
9	18	0.31205	0.33985	5.7600
9	19	0.34464	0.35968	6.0800
9	20	0.35491	0.37944	6.4000

FOR ALPHA=.05 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
9	9	0.69714	0.87890	18.0000
9	10	0.76796	0.91562	20.0000
9	11	0.83089	0.94228	22.0000
9	12	0.82115	0.96118	24.0000
9	13	0.85552	0.97430	26.0000
9	14	0.90792	0.98323	28.0000
9	15	0.92573	1.00000	30.0000
9	16	0.93714	1.00000	32.0000
9	17	0.94233	1.00000	34.0000
9	18	0.95873	1.00000	36.0000
9	19	0.96893	1.00000	38.0000
9	20	0.97043	1.00000	40.0000

FOR ALPHA=.05 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
9	9	0.99108	1.00000	46.0800
9	10	0.99688	1.00000	51.2000
9	11	0.99870	1.00000	56.3200
9	12	0.99884	1.00000	61.4400
9	13	0.99956	1.00000	66.5600
9	14	0.99988	1.00000	71.6799
9	15	0.99993	1.00000	76.7999
9	16	0.99996	1.00000	81.9200
9	17	0.99998	1.00000	87.0400
9	18	0.99999	1.00000	92.1600
9	19	1.00000	1.00000	97.2800
9	20	1.00000	1.00000	102.4000

TABLE 15. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.01 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
9	9	0.06347	0.05558	2.8800
9	10	0.07427	0.06411	3.2000
9	11	0.08245	0.07316	3.5200
9	12	0.10[illegible]4	0.08273	3.8400
9	13	0.0[illegible]7	0.09283	4.1600
9	14	0.109[illegible]3	0.10344	4.4800
9	15	0.139[illegible]9	0.11455	4.8000
9	16	0.13377	0.12616	5.1200
9	17	0.14662	0.13824	5.4400
9	18	0.15051	0.15077	5.7600
9	19	0.18440	0.16375	6.0800
9	20	0.16744	0.17713	6.4000

FOR ALPHA=.01 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
9	9	0.52995	0.71478	18.0000
9	10	0.58859	0.78115	20.0000
9	11	0.66749	0.83542	22.0000
9	12	0.74554	0.87853	24.0000
9	13	0.73779	1.00000	26.0000
9	14	0.78479	1.00000	28.0000
9	15	0.84614	1.00000	30.0000
9	16	0.84667	1.00000	32.0000
9	17	0.87249	1.00000	34.0000
9	18	0.89675	1.00000	36.0000
9	19	0.92633	1.00000	38.0000
9	20	0.93173	1.00000	40.0000

FOR ALPHA=.01 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
9	9	0.96885	1.00000	46.0800
9	10	0.98457	1.00000	51.2000
9	11	0.99414	1.00000	56.3200
9	12	0.99749	1.00000	61.4400
9	13	0.99780	1.00000	66.5600
9	14	0.99912	1.00000	71.6799
9	15	0.99973	1.00000	76.7999
9	16	0.99974	1.00000	81.9200
9	17	0.99989	1.00000	87.0400
9	18	0.99996	1.00000	92.1600
9	19	0.99998	1.00000	97.2800
9	20	0.99999	1.00000	102.4000

TABLE 16. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.05 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
10	10	0.17527	0.15522	3.6000
10	11	0.23055	0.17343	3.9600
10	12	0.21809	0.19210	4.3200
10	13	0.26116	0.21119	4.6800
10	14	0.25499	0.23065	5.0400
10	15	0.27928	0.25043	5.4000
10	16	0.28559	0.27050	5.7600
10	17	0.29710	0.29079	6.1200
10	18	0.32143	0.31126	6.4800
10	19	0.34939	0.33184	6.8400
10	20	0.37842	0.35249	7.2000

FOR ALPHA=.05 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
10	10	0.77456	0.92379	22.5000
10	11	0.84985	0.94983	24.7500
10	12	0.85073	0.96763	27.0000
10	13	0.89036	1.00000	29.2500
10	14	0.90764	1.00000	31.5000
10	15	0.93158	1.00000	33.7500
10	16	0.93771	1.00000	36.0000
10	17	0.94783	1.00000	38.2500
10	18	0.96239	1.00000	40.5000
10	19	0.97304	1.00000	42.7500
10	20	0.97983	1.00000	45.0000

FOR ALPHA=.05 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
10	10	0.99701	1.00000	57.6000
10	11	0.99915	1.00000	63.3600
10	12	0.99920	1.00000	69.1200
10	13	0.99971	1.00000	74.8800
10	14	0.99988	1.00000	80.6400
10	15	0.99995	1.00000	86.4000
10	16	0.99997	1.00000	92.1600
10	17	0.99999	1.00000	97.9200
10	18	0.99999	1.00000	103.6800
10	19	1.00000	1.00000	109.4400
10	20	1.00000	1.00000	115.2000

TABLE 17. A COMPARISON OF THE EXACT AND ASYMPTOTIC POWER

FOR ALPHA=.01 AND RHO=.80

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
10	10	0.10446	0.05117	3.6000
10	11	0.09642	0.05932	3.9600
10	12	0.10035	0.06809	4.3200
10	13	0.11510	0.07749	4.6800
10	14	0.14006	0.08750	5.0400
10	15	0.12472	0.09814	5.4000
10	16	0.14994	0.10937	5.7600
10	17	0.16676	0.12119	6.1200
10	18	0.18094	0.13358	6.4800
10	19	0.17739	0.14651	6.8400
10	20	0.21099	0.15997	7.2000

FOR ALPHA=.01 AND RHO=.50

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
10	10	0.68188	0.79805	22.5000
10	11	0.67434	1.00000	24.7500
10	12	0.74832	1.00000	27.0000
10	13	0.77276	1.00000	29.2500
10	14	0.82566	1.00000	31.5000
10	15	0.83740	1.00000	33.7500
10	16	0.87995	1.00000	36.0000
10	17	0.89731	1.00000	38.2500
10	18	0.91428	1.00000	40.5000
10	19	0.92740	1.00000	42.7500
10	20	0.94904	1.00000	45.0000

FOR ALPHA=.01 AND RHO=.20

K CLASSES	N OBSERVATIONS	EXACT POWER	ASYMPT POWER	LAMBDA
10	10	0.99187	1.00000	57.6000
10	11	0.99435	1.00000	63.3600
10	12	0.99803	1.00000	69.1200
10	13	0.99846	1.00000	74.8800
10	14	0.99942	1.00000	80.6400
10	15	0.99971	1.00000	86.4000
10	16	0.99989	1.00000	92.1600
10	17	0.99993	1.00000	97.9200
10	18	0.99997	1.00000	103.6800
10	19	0.99999	1.00000	109.4400
10	20	0.99999	1.00000	115.2000

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C  TITLE:  COMPARITIVE PROGRAM FOR EVALUATING THE EXACT AND ASYMPTOTIC
C          POWERS OF THE CHI-SQUARE GOODNESS-OF-FIT TEST
C
C  PROGRAMMER:  BRIAN T. WRIGHT, OPERATIONS ANALYSIS CIRRICULUM, NAVAL
C               POSTGRADUATE SCHOOL, MONTEREY, CALIFORNIA.
C
C  DATE PREPARED:  1DECEMBER 1970
C
C  PURPOSE:
C  THIS PROGRAM WAS DEVELOPED TO GENERATE MULTINOMIAL PROBABILITIES FOR
C  CLASSES IN THE RANGE 3 THROUGH 10.  THE PROGRAM HAS THE CAPABILITY OF
C  HANDLING OBSERVATIONS OF THE RANGE K THROUGH 20 WHERE K IS THE NUMBER
C  OF MULTINOMIAL CLASSES FOR EACH OF THE MULTINOMIAL DISTRIBUTIONS.
C
C  THE DIMENSIONED ARRAYS ARE AS FOLLOWS; N=AN ARRAY FOR THE GENERATED
C  PARTITION, K= A WORKING ARRAY DURING THE GENERATION PROCESS, NF IS AN
C  ARRAY TO STORE THE FACTORIAL PRODUCTS CORRESPONDING TO THE GENERATED
C  PARTITION, AND KC IS USED FOR DETERMINING THE COEFFIENT OF OCCURRANCE.
C
      REAL*8 BETA,PPAR,PTCT,P,PB,PI,CR,BA2,GB
      REAL*4 MC,NF,JF
      INTEGER*4 KCC,KPD,KNP
      DIMENSION N(10),K(10),NF(10),KC(20)
C
C  THE FOLLOWING DATA MUST BE SUPPLIED TO THE PROGRAM
C    KK= THE NUMBER OF CLASSES UNDER CONSIDERATION
C    NN= THE STARTING NUMBER OF OBSERVATIONS (MUST BE LESS THAN 20)
C    Y=0 THE NON-CENTRALITY PARAMETER (TO BE USED ONLY IF THIS PROGRAM
C          IS INCORPORATED AS A SUBROUTINE AND THEN AS A RETURN)
C    C=THE CRITICAL POINT OF THE CHI-SQUARE TEST FOR (KK-1) DEGREES OF
C        FREEDOM AND THE ALPHA SPECIFIED IN A BELOW
C    A= THE ALPHA LEVEL FOR THE CHI-SQUARE TEST
C    B= A PARAMETER TO BE USED FOR  THE EXACT POWER OF THE TEST THIS IS
C         A RETURN PARAMETER IF A SUBROUTINE IS USED
C
C **** WARNING *** THIS PROGRAM IS ONLY VALID FOR UP TO 20 OBSERVATIONS
C                  IF MORE ARE REQUIRED CHECK ALL INTEGER VARIABLES TO
C                  INSURE THEY ARE CAPABLE OF BEING HANDLED BY THE COMP-
C                  TER.
C
      CALL ERRSET(208,256,-1,1,0,207)
 1111 READ(5,2) NN,KK,Y,C,A,B
    2 FORMAT(2I3,2F8.5,2F6.4)
      WRITE(6,1000)
 1000 FORMAT('1',////,14X,'TABLE      A COMPARISON OF THE EXACT AND ASYMP
     1TOTIC POWER')
      DO 1 IL=1,3
      GO TO (4001,4002,4003),IL
```

```
4001 X=.8
     GO TO 4004
4002 X=.5
     GO TO 4004
4003 X=.2
4004 PTOT=0.0
     DO 999 I=1,10
     K(I)=0
 999 N(I)=0
     JJ=KK-2
     Y=X
     NDIM=10
     WRITE(6,1100) A,X
1100 FORMAT('0',26X,'FOR ALPHA=',F3.2,'  AND RHO=',F3.2,//,14X,'K CLASS
    1ES N OBSERVATIONS  EXACT POWER  ASYMPT POWER LAMBDA'/)
     NN=KK
     GO TO (30,40,50,60,70,80,90,120,990),JJ
C
C  THE BELOW GENERATES PARTITIONS FOR A MULTINOMIAL OF 3 CLASSES
C
  30 N(1)=NN
     CALL AMSP(C,KK,NN,X,BA2,YMB)
     INDEX=NN/KK
  31 IF(N(1).LE.INDEX) GO TO 38
     K(1)=NN-N(1)
     IF(K(1).GT.N(1)) K(1)=N(1)
     K(2)=NN-N(1)-K(1)
     IF(K(2).GT.K(1)) GO TO 35
     CALL RITE(N,K,NF,NDIM,NN,KK,Y,C,KC,A,B,PTOT,&33)
  33 DO 32 I=1,NN
     K(1)=K(1)-1
     K(2)=K(2)+1
     IF(K(2).GT.K(1)) GO TO 35
     CALL RITE(N,K,NF,NDIM,NN,KK,Y,C,KC,A,B,PTOT,&32)
  32 CONTINUE
  35 N(1)=N(1)-1
     GO TO 31
  38 NN=NN+1
     NK=NN-1
     BETA=1.0-PTOT
     WRITE(6,1001) KK,NK,PTOT,BA2,YMB
1001 FORMAT(14X,I5,10X,I3,8X,F8.5,5X,F8.5,3X,F8.4)
C **
     IF(NN.EQ.21) GO TO 1
     PTOT=0.0
     GO TO 30
C
C  THE BELOW GENERATES PARTITIONS FOR A MULTINOMIAL OF 4 CLASSES
```

```
C
   40 N(1)=NN
      CALL AMSP(C,KK,NN,X,BA2,YMB)
      INDEX=NN/KK
   41 IF(N(1).LE.INDEX) GO TO 48
      K(1)=NN-N(1)
      IF(K(1).GT.N(1)) K(1)=N(1)
      K(2)=NN-N(1)-K(1)
   42 IF(K(2).GT.K(1)) K(2)=K(1)
      K(3)=NN-N(1)-K(1)-K(2)
   43 IF(K(3).GT.K(2)) GO TO 45
      CALL RITE(N,K,NF,NDIM,NN,KK,Y,C,KC,A,B,PTOT,&44)
   44 IF(K(2).GT.1) GO TO 46
 4444 K(1)=K(1)-1
      K(2)=NN-N(1)-K(1)
      GO TO 42
   46 K(2)=K(2)-1
      K(3)=NN-N(1)-K(1)-K(2)
      GC TO 43
   45 IF(K(1).GT.K(3)) GO TO 4444
      N(1)=N(1)-1
      GO TO 41
   48 NN=NN+1
      BETA=1.0-PTOT
      NK=NN-1
      WRITE(6,1001) KK,NK,PTOT,BA2,YMB
C **
      IF(NN.EQ.21) GO TO 1
      PTOT=0.0
      GO TO 40
C
C  THE BELOW GENERATES PARTITIONS FOR A MULTINOMIAL OF 5 CLASSES
C
   50 N(1)=NN
      CALL AMSP(C,KK,NN,X,BA2,YMB)
      INDEX=NN/KK
   51 IF(N(1).LE.INDEX) GO TO 58
      K(1)=NN-N(1)
      IF(K(1).GT.N(1)) K(1)=N(1)
      K(2)=NN-N(1)-K(1)
   52 IF(K(2).GT.K(1)) K(2)=K(1)
      K(3)=NN-N(1)-K(1)-K(2)
   53 IF(K(3).GT.K(2)) K(3)=K(2)
      K(4)=NN-N(1)-K(1)-K(2)-K(3)
   54 IF(K(4).GT.K(3)) GO TO 55
      CALL RITE(N,K,NF,NDIM,NN,KK,Y,C,KC,A,B,PTOT,&56)
   56 IF(K(3).GT.1) GO TO 59
      IF(K(2).GT.1) GO TO 57
```

```
  556 K(1)=K(1)-1
      K(2)=NN-N(1)-K(1)
      GO TO 52
   57 K(2)=K(2)-1
      K(3)=NN-N(1)-K(1)-K(2)
      GO TO 53
   59 K(3)=K(3)-1
      K(4)=NN-N(1)-K(1)-K(2)-K(3)
      GO TO 54
   55 IF(K(2).GT.K(4)) GO TO 57
      IF(K(1).GT.K(4)) GO TO 556
      N(1)=N(1)-1
      GO TO 51
   58 NN=NN+1
      BETA=1.0-PTOT
      NK=NN-1
      WRITE(6,1001) KK,NK,PTOT,BA2,YMB
C **
      IF(NN.EQ.21) GO TO 1
      PTOT=0.0
      GO TO 50
C
C  THE BELOW GENERATES PARTITIONS FOR A MULTINOMIAL OF 6 CLASSES
C
   60 N(1)=NN
      CALL AMSP(C,KK,NN,X,BA2,YMB)
      INDEX=NN/KK
   61 IF(N(1).LE.INDEX) GO TO 69
      K(1)=NN-N(1)
      IF(K(1).GT.N(1)) K(1)=N(1)
      K(2)=NN-N(1)-K(1)
  666 IF(K(2).GT.K(1)) K(2)=K(1)
      K(3)=NN-N(1)-K(1)-K(2)
  667 IF(K(3).GT.K(2)) K(3)=K(2)
      K(4)=NN-N(1)-K(2)-K(3)-K(1)
  668 IF(K(4).GT.K(3)) K(4)=K(3)
      K(5)=NN-N(1)-K(1)-K(2)-K(3)-K(4)
  669 IF(K(5).GT.K(4)) GO TO 66
      CALL RITE(N,K,NF,NDIM,NN,KK,Y,C,KC,A,B,PTOT,&676)
  676 IF(K(4).GT.1) GO TO 674
      IF(K(3).GT.1) GO TO 672
      IF(K(2).GT.1) GO TO 670
  677 K(1)=K(1)-1
      K(2)=NN-N(1)-K(1)
      GO TO 666
  670 K(2)=K(2)-1
      K(3)=NN-N(1)-K(1)-K(2)
      GO TO 667
```

```
  672 K(3)=K(3)-1
      K(4)=NN-N(1)-K(1)-K(2)-K(3)
      GO TO 668
  674 K(4)=K(4)-1
      K(5)=NN-N(1)-K(1)-K(2)-K(3)-K(4)
      GO TO 669
   66 IF(K(3).GT.K(5)) GO TO 672
      IF(K(2).GT.K(5)) GO TO 670
      IF(K(1).GT.K(5)) GO TO 677
      N(1)=N(1)-1
      GO TO 61
   69 NN=NN+1
C **
      BETA=1.0-PTOT
      NK=NN-1
      WRITE(6,1001) KK,NK,PTOT,BA2,YMB
      IF(NN.EQ.21) GO TO 1
      PTOT=0.0
      GO TO 60
C
C  THE BELOW GENERATES PARTITIONS FOR A MULTINOMIAL OF 7 CLASSES
C
   70 N(1)=NN
      CALL AMSP(C,KK,NN,X,BA2,YMB)
      INDEX=NN/KK
   71 IF(N(1).LE.INDEX) GO TO 79
      K(1)=NN-N(1)
      IF(K(1).GT.N(1)) K(1)=N(1)
      K(2)=NN-N(1)-K(1)
   72 IF(K(2).GT.K(1)) K(2)=K(1)
      K(3)=NN-N(1)-K(1)-K(2)
   73 IF(K(3).GT.K(2)) K(3)=K(2)
      K(4)=NN-N(1)-K(2)-K(3)-K(1)
   74 IF(K(4).GT.K(3)) K(4)=K(3)
      K(5)=NN-N(1)-K(1)-K(2)-K(3)-K(4)
   75 IF(K(5).GT.K(4)) K(5)=K(4)
      K(6)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)
   76 IF(K(6).GT.K(5)) GO TO 77
      CALL RITE(N,K,NF,NDIM,NN,KK,Y,C,KC,A,B,PTOT,&701)
  701 IF(K(5).GT.1) GO TO 705
      IF(K(4).GT.1) GO TO 704
      IF(K(3).GT.1) GO TO 703
      IF(K(2).GT.1) GO TO 702
  707 K(1)=K(1)-1
      K(2)=NN-N(1)-K(1)
      GO TO 72
  702 K(2)=K(2)-1
      K(3)=NN-N(1)-K(1)-K(2)
```

```
      GO TO 73
  703 K(3)=K(3)-1
      K(4)=NN-N(1)-K(1)-K(2)-K(3)
      GO TO 74
  704 K(4)=K(4)-1
      K(5)=NN-N(1)-K(1)-K(2)-K(3)-K(4)
      GO TO 75
  705 K(5)=K(5)-1
      K(6)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)
      GO TO 76
   77 IF(K(4).GT.K(6)) GO TO 704
      IF(K(3).GT.K(6)) GO TO 703
      IF(K(2).GT.K(6)) GO TO 702
      IF(K(1).GT.K(6)) GO TO 707
      N(1)=N(1)-1
      GO TO 71
   79 NN=NN+1
      BETA=1.0-PTOT
      NK=NN-1
      WRITE(6,1001) KK,NK,PTOT,BA2,YMB
C **
      IF(NN.EQ.21) GO TO 1
      PTOT=0.0
      GO TO 70
C
C  THE BELOW GENERATES PARTITIONS FOR A MULTINOMIAL OF 8 CLASSES
C
   80 N(1)=NN
      CALL AMSP(C,KK,NN,X,BA2,YMB)
      INDEX=NN/KK
   81 IF(N(1).LE.INDEX) GO TO 89
      K(1)=NN-N(1)
      IF(K(1).GT.N(1)) K(1)=N(1)
      K(2)=NN-N(1)-K(1)
   82 IF(K(2).GT.K(1)) K(2)=K(1)
      K(3)=NN-N(1)-K(1)-K(2)
   83 IF(K(3).GT.K(2)) K(3)=K(2)
      K(4)=NN-N(1)-K(1)-K(2)-K(3)
   84 IF(K(4).GT.K(3)) K(4)=K(3)
      K(5)=NN-N(1)-K(1)-K(2)-K(3)-K(4)
   85 IF(K(5).GT.K(4)) K(5)=K(4)
      K(6)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)
   86 IF(K(6).GT.K(5)) K(6)=K(5)
      K(7)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)
   88 IF(K(7).GT.K(6)) GO TO 87
      CALL RITE(N,K,NF,NDIM,NN,KK,Y,C,KC,A,B,PTOT,&809)
  809 IF(K(6).GT.1) GO TO 805
      IF(K(5).GT.1) GO TO 804
```

```
      IF(K(4).GT.1) GO TO 803
      IF(K(3).GT.1) GO TO 802
      IF(K(2).GT.1) GO TO 801
  810 K(1)=K(1)-1
      K(2)=NN-N(1)-K(1)
      GO TO 82
  801 K(2)=K(2)-1
      K(3)=NN-N(1)-K(1)-K(2)
      GO TO 83
  802 K(3)=K(3)-1
      K(4)=NN-N(1)-K(1)-K(2)-K(3)
      GO TO 84
  803 K(4)=K(4)-1
      K(5)=NN-N(1)-K(1)-K(2)-K(3)-K(4)
      GO TO 85
  804 K(5)=K(5)-1
      K(6)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)
      GO TO 86
  805 K(6)=K(6)-1
      K(7)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)
      GO TO 88
   87 IF(K(5).GT.K(7)) GO TO 804
      IF(K(4).GT.K(7)) GO TO 803
      IF(K(3).GT.K(7)) GO TO 802
      IF(K(2).GT.K(7)) GO TO 801
      IF(K(1).GT.K(7)) GO TO 810
      N(1)=N(1)-1
      GO TO 81
   89 NN=NN+1
      BETA=1.0-PTOT
      NK=NN-1
      WRITE(6,1001) KK,NK,PTOT,BA2,YMB
C **
      IF(NN.EQ.21) GO TO 1
      PTOT=0.0
      GO TO 80
C
C  THE BELOW GENERATES PARTITIONS FOR A MULTINOMIAL OF 9 CLASSES
C
   90 N(1)=NN
      CALL AMSP(C,KK,NN,X,BA2,YMB)
      INDEX=NN/KK
   91 IF(N(1).LE.INDEX) GO TO 199
      K(1)=NN-N(1)
      IF(K(1).GT.N(1)) K(1)=N(1)
      K(2)=NN-N(1)-K(1)
   92 IF(K(2).GT.K(1)) K(2)=K(1)
      K(3)=NN-N(1)-K(1)-K(2)
```

```
   93 IF(K(3).GT.K(2)) K(3)=K(2)
      K(4)=NN-N(1)-K(1)-K(2)-K(3)
   94 IF(K(4).GT.K(3)) K(4)=K(3)
      K(5)=NN-N(1)-K(1)-K(2)-K(3)-K(4)
   95 IF(K(5).GT.K(4)) K(5)=K(4)
      K(6)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)
   96 IF(K(6).GT.K(5)) K(6)=K(5)
      K(7)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)
   97 IF(K(7).GT.K(6)) K(7)=K(6)
      K(8)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)-K(7)
   98 IF(K(8).GT.K(7)) GO TO 197
      CALL RITE(N,K,NF,NDIM,NN,KK,Y,C,KC,A,B,PTOT,&101)
  101 IF(K(7).GT.1) GO TO 107
      IF(K(6).GT.1) GO TO 106
      IF(K(5).GT.1) GO TO 105
      IF(K(4).GT.1) GO TO 104
      IF(K(3).GT.1) GO TO 103
      IF(K(2).GT.1) GO TO 102
 1004 K(1)=K(1)-1
      K(2)=NN-N(1)-K(1)
      GO TO 92
  102 K(2)=K(2)-1
      K(3)=NN-N(1)-K(1)-K(2)
      GO TO 93
  103 K(3)=K(3)-1
      K(4)=NN-N(1)-K(1)-K(2)-K(3)
      GO TO 94
  104 K(4)=K(4)-1
      K(5)=NN-N(1)-K(1)-K(2)-K(3)-K(4)
      GO TO 95
  105 K(5)=K(5)-1
      K(6)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)
      GO TO 96
  106 K(6)=K(6)-1
      K(7)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)
      GO TO 97
  107 K(7)=K(7)-1
      K(8)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)-K(7)
      GO TO 98
  197 IF(K(6).GT.K(8)) GO TO 106
      IF(K(5).GT.K(8)) GO TO 105
      IF(K(4).GT.K(8)) GO TO 104
      IF(K(3).GT.K(8)) GO TO 103
      IF(K(2).GT.K(8)) GO TO 102
      IF(K(1).GT.K(8)) GO TO 1004
      N(1)=N(1)-1
      GO TO 91
  199 NN=NN+1
```

```
      BETA=1.0-PTOT
      NK=NN-1
      WRITE(6,1001) KK,NK,PTOT,BA2,YMB
C **
      IF(NN.EQ.21) GO TO 1
      PTOT=0.0
      GO TO 90
C
C  THE BELOW GENERATES PARTITIONS FOR A MULTINOMIAL OF 10 CLASSES
C
  120 N(1)=NN
      CALL AMSP(C,KK,NN,X,BA2,YMB)
      INDEX=NN/KK
  121 IF(N(1).LE.INDEX) GO TO 139
      K(1)=NN-N(1)
      IF(K(1).GT.N(1)) K(1)=N(1)
      K(2)=NN-N(1)-K(1)
  122 IF(K(2).GT.K(1)) K(2)=K(1)
      K(3)=NN-N(1)-K(1)-K(2)
  123 IF(K(3).GT.K(2)) K(3)=K(2)
      K(4)=NN-N(1)-K(1)-K(2)-K(3)
  124 IF(K(4).GT.K(3)) K(4)=K(3)
      K(5)=NN-N(1)-K(1)-K(2)-K(3)-K(4)
  125 IF(K(5).GT.K(4)) K(5)=K(4)
      K(6)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)
  126 IF(K(6).GT.K(5)) K(6)=K(5)
      K(7)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)
  127 IF(K(7).GT.K(6)) K(7)=K(6)
      K(8)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)-K(7)
  128 IF(K(8).GT.K(7)) K(8)=K(7)
       K(9)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)-K(7)-K(8)
  129 IF(K(9).GT.K(8)) GO TO 157
      CALL RITE(N,K,NF,NDIM,NN,KK,Y,C,KC,A,B,PTOT,&130)
  130 IF(K(8).GT.1) GO TO 138
      IF(K(7).GT.1) GO TO 137
      IF(K(6).GT.1) GO TO 136
      IF(K(5).GT.1) GO TO 135
      IF(K(4).GT.1) GO TO 134
      IF(K(3).GT.1) GO TO 133
      IF(K(2).GT.1) GO TO 132
  131 K(1)=K(1)-1
      K(2)=NN-N(1)-K(1)
      GO TO 122
  132 K(2)=K(2)-1
      K(3)=NN-N(1)-K(1)-K(2)
      GO TO 123
  133 K(3)=K(3)-1
      K(4)=NN-N(1)-K(1)-K(2)-K(3)
```

```
      GO TO 124
  134 K(4)=K(4)-1
      K(5)=NN-N(1)-K(1)-K(2)-K(3)-K(4)
      GO TO 125
  135 K(5)=K(5)-1
      K(6)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)
      GO TO 126
  136 K(6)=K(6)-1
      K(7)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)
      GO TO 127
  137 K(7)=K(7)-1
      K(8)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)-K(7)
      GO TO 128
  138 K(8)=K(8)-1
       K(9)=NN-N(1)-K(1)-K(2)-K(3)-K(4)-K(5)-K(6)-K(7)-K(8)
      GO TO 129
  157 IF(K(7).GT.K(9)) GO TO 137
      IF(K(6).GT.K(9)) GO TO 136
      IF(K(5).GT.K(9)) GO TO 135
      IF(K(4).GT.K(9)) GO TO 134
      IF(K(3).GT.K(9)) GO TO 133
      IF(K(2).GT.K(9)) GO TO 132
      IF(K(1).GT.K(9)) GO TO 131
      N(1)=N(1)-1
      GO TO 121
  139 NN=NN+1
      BETA=1.0-PTOT
      NK=NN-1
      WRITE(6,1001) KK,NK,PTOT,BA2,YMB
C **
      IF(NN.EQ.21) GO TO 1
      PTOT=0.0
      GO TO 120
  990 STOP
    1 CONTINUE
      GO TO 1111
      END


      SUBROUTINE RITE(N,K,NF,NDIM,NN,KK,Y,C,KC,A,B,PTOT,*)
      REAL*8 BETA,PPAR,PTOT,P,PB,PI,CR,BA2,GB
      REAL*4 MC,NF,JF
      INTEGER*4 KOC,KPD,KNP
C
C   THIS SUBROUTINE COMPUTES THE MULTINOMIAL PROBABILITY ASSOCIATED
C   WITH THE GENERATED PARTITIONS AND SUMS THE PROBABILITIES WHICH ARE
C   GREATER THAN THE CRITICAL POINT AS SPECIFIED BY THE CHI-SQUARE TEST
C
```

```
      DIMENSION N(NDIM),K(NDIM),NF(NDIM),KC(NN)
      KN=0
      KI=KK-1
C
C     LCAD PARTITION IN ARRAY N
C
      DO 99 I=1,KI
   99 N(I+1)=K(I)
C
C     DETERMINE THE VALUE OF THE SUM OF SQUARES
C
      DO 100 I=1,KK
  100 KN=KN+N(I)**2
C
C     CCMPUTE CRITICAL REGION FROM C OF CENTRAL CHI-SQUARE
C
      CD=FLOAT(NN)
      CF=(CD+C)*(CD/KK)
      IC=IFIX(CF)
C
C     IF THE SUM OF SQUARES IS LESS THAN OR EQUAL TO THE GREATEST INTEGER
C     OF THE COMPUTED CRITICAL REGION RETURN TO MAIN PROGRAM AND GENERATE A
C     NEW PARTITION.
C
      IF(KN.LE.IC) RETURN 1
      PI=Y/KK
      PB=1.0-KI*PI
C
C     COMPUTE N
C
      FCT=1.0
      DO 102 I=1,NN
  102 FCT=FCT*I
C
C     COMPUTE N(I)  AND LOAD INTO NF
C
      DO 103 J=1,KK
      FCTJ=1.0
      IF(N(J).LE.1) GO TO 105
      NNN=N(J)
      DO 104 I=1,NNN
  104 FCTJ=FCTJ*I
      NF(J)=FCTJ
  103 CONTINUE
      GO TO 110
  105 NF(J)=1.0
      GO TO 103
  110 FCTT=1.0
```

```
      DO 106 J=1,KK
  106 FCTT=FCTT*NF(J)
C
C  COMPUTE THE MULTINOMIAL COEFFICIENT.
C
      MC=FCT/FCTT
      PPAR=0.0
      NCHK=0
  111 DO 107 I=1,KK
      IF(N(I).EQ.NCHK) GO TO 107
      NX=NN-N(I)
C
C  COMPUTE COEFFICIENT OF OCCURRANCE.
C
      DO 115 J=1,NN
  115 KC(J)=0
      KCSM=0
      KPD=1
      DO 120 J=1,KK
      DO 120 JJ=1,NN
      IF(J.EQ.I) GO TO 120
      IF(N(J).EQ.JJ) KC(JJ)=KC(JJ)+1
  120 CONTINUE
      DO 121 J=1,NN
  121 KCSM=KCSM+KC(J)
      KCO=KI-KCSM
      IF(KCO.LE.1) GO TO 124
      DO 122 J=1,KCO
  122 KPD=KPD*J
  124 DO 123 J=1,NN
      IF(KC(J).LE.1) GO TO 123
      KCN=KC(J)
      DO 1230 JJ=1,KCN
      KPD=KPD*JJ
 1230 CONTINUE
  123 CONTINUE
      KNP=1
      DO 125 J=1,KI
  125 KNP=KNP*J
      KOC=KNP/KPD
      P=DABS(MC*KOC*(PI**NX)*(PB**N(I)))
       PPAR=PPAR+P
      NCHK=N(I)
  107 CONTINUE
C
C  SUM ALL PROBABILITIES EXCEEDING THE SPECIFIED CRITICAL POINT
C
      PTOT=PTOT+PPAR
```

```
      RETURN 1
      END

      SUBROUTINE AMSP(C,KK,NN,X,BA2,YMB)
C
C
C    THIS SUBROUTINE COMPUTES THE ASYMPTOTIC POWER BASED ON A METHOD BY
C     E. FIX.
C
      REAL*8 BETA,PPAR,PTOT,P,PB,PI,CR,BA2,GB
      REAL*4 MC,NF,JF
      INTEGER*4 KOC,KPD,KNP
      CR=C/2.0
      ITST=0
      BA2=0.0
      YMB=NN*(KK-1)*(X-1.0)**2/KK+NN*((KK-1)*(1.0-X))**2/KK
      XP=EXP(-YMB/2)
      DO 100 J=1,50
      JF=1.0
      IST=J
      DO 101 JJ=1,IST
  101 JF=JF*JJ
      AA=DFLOAT((KK-1)/2+J)
      CALL GAMA(AA,CR,GAM,GB,ER)
      BA2=BA2+XP*((YMB/2)**J)*GAM/JF
      TST=BA2*1000000
      JCHK=IFIX(TST)
      IF(JCHK.EQ.ITST) GO TO 102
      ITST=JCHK
  100 CONTINUE
  102 IF(JCHK.EQ.0) BA2=1.0
      RETURN
      END
C     ..................................................................GAMA0000
C                                                                       GAMA0010
C                                                                       GAMA0020
C  A. IDENTIFICATION                                                    GAMA0030
C     TITLE: NORMALIZED INCOMPLETE GAMMA FUNCTION WITH POISSON TERM     GAMA0040
C     SHARE ID: C3-UR-GAMA (FORTRAN IV FOR THE IBM SYSTEM/360)          GAMA0050
C     PROGRAMMER: JOHN R.B. WHITTLESEY, DEPARTMENT OF PSYCHIATRY        GAMA0060
C                 UCLA MEDICAL CENTER, NEURO-PSYCHIATRIC INSTITUTE      GAMA0070
C                 LOS ANGELES, CALIFORNIA                               GAMA0080
C     DATE: 2 SEPTEMBER 1961; CHECKED OUT AT NPG BY D.CHACE OCT 1968    GAMA0090
C                                                                       GAMA0100
C  B. PURPOSE                                                           GAMA0110
C     TO EVALUATE THE NORMALIZED INCOMPLETE GAMMA FUNCTION,             GAMA0120
C                                                                       GAMA0130
```

```
C     GAM(A,X)=(1./GAMMA(A)) TIMES THE INTEGRAL OF EXP(-U)*U**(A-1.).    GAMA0140
C                                FOR 'U' BETWEEN 'X' AND PLUS INFINITYGAMA0150
C     AND ITS ASSOCIATED POISSON TERM,                                   GAMA0160
C                                                                        GAMA0170
C                  B = EXP(-X)*X**A/GAMMA(A+1) = GAM(A+1,X)-GAM(A,X)GAMA0180
C                                                                        GAMA0190
C  C. USAGE                                                              GAMA0200
C     1. CALLING SEQUENCE:                                               GAMA0210
C                        CALL GAMA(A,X,GAM,B,ER)                         GAMA0220
C     2. ARGUMENTS:                                                      GAMA0230
C        A - REAL*8, GREATER THAN OR EQUAL TO ZERO. (SEE SECTION B)      GAMA0240
C        X - REAL*8, GREATER THAN OR EQUAL TO ZERO. (SEE SECTION B)      GAMA0250
C        GAM - GAM=1 IF 'X' NOT ZERO AND 'A' EQUAL TO OR LESS THAN ZERO.GAMA0260
C              GAM=0 WHENEVER X=0 IRRESPECTIVE OF 'A'. (REAL*8 RESULT)   GAMA0270
C        B - REAL*8, THE POISSON TERM. (SEE SECTION B)                   GAMA0280
C        ER - EQUAL TO ZERO, NORMAL RETURN.                              GAMA0290
C           - GREATER THAN ZERO, ERROR RETURN. (ER IS REAL*8)            GAMA0300
C                                                                        GAMA0310
C  D. SUBROUTINE SUPPORT                                                 GAMA0320
C     IN THE PACKAGE FOR SUBROUTINE 'GAMA', THE SUBROUTINES 'WHICH',     GAMA0330
C     'GAUSS', AND 'GAMMA' ARE INCLUDED. THESE NAMES COULD CONFLICT WITHGAMA0340
C     OTHER ROUTINES INCLUDED WITH THE USER'S PROGRAM OR WHICH HE USES   GAMA0350
C     FROM FORTLIB. IF SO, USER MUST ALTER ONE OR MORE SUBROUTINE NAMES.GAMA0360
C     THE LIBRARY FUNCTIONS 'INT', 'SQRT', 'LOG', 'ABS', AND 'EXP' MUST  GAMA0370
C     BE AVAILABLE.                                                      GAMA0380
C                                                                        GAMA0390
C  E. ACCURACY                                                           GAMA0400
C     IN GENERAL, THE ABSOLUTE ERROR IS LESS THAN .000001 FOR 'A' LESS   GAMA0410
C     THAN 100., 'A' GREATER THAN .1, FURTHER, THE RELATIVE ERROR IN     GAMA0420
C     THE SMALLER OF GAM(A,X) AND 1.-GAM(A.X) IS USUALLY ONLY A FEW      GAMA0430
C     UNITS IN THE 7TH SIGNIFICANT FIGURE FOR GAM NOT EQUAL TO ZERO.     GAMA0440
C     HOWEVER, FOR 'X' NEAR 'A', AND 'A' GREATER THAN 100, THE ERROR MAYGAMA0450
C     GROW TO SEVERAL UNITS IN THE 6TH SIGNIFICANT FIGURE, OR EVEN THE   GAMA0460
C     5TH, AS 'A' BECOMES VERY LARGE.                                    GAMA0470
C                                                                        GAMA0480
C     ..................................................................GAMA0490

      SUBROUTINE GAMA (A,X,GAM,B,ER )                                    GAMA0000
C                                                                        GAMA0010
C     THE PURPOSE OF THIS SUBROUTINE IS TO CALL GAMMA(X,A,GAM,B,NVB)     GAMA0020
C     WHICH IS THE REAL BUSINESS END OF THE NORMALIZED INCOMPLETE GAMMA  GAMA0030
C     FUNCTION EVALUATING ROUTINE.                                       GAMA0040
C                                                                        GAMA0050
C     THIS SUBROUTINE MAY BE ALTERED - WITHOUT CHANGING GAMMA(      )    GAMA0060
C     E.G. BY DIVIDING THE PARAMETERS BY TWO SO AS TO GET CHI-SQUARES    GAMA0070
C                                                                        GAMA0080
C     GAM=1 IF X IS NOT ZERO AND A=0 OR A NEGATIVE INTEGER               GAMA0090
```

```
C     GAM=0 WHENEVER X=0 IRRESPECTIVE OF A
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION NVB(5)
      ER=0.DO
      IF(X) 35,43,27
   27 IF(A) 28,28,43
   28 IF(A-IDINT(A)) 35,37,37
   35 ER=2.DO
   37 GAM=0.DO
      B=0.DO
      RETURN
   43 CALL GAMMA(A,X,GAM,B,NVB )
C
      IF(NVB(2)-4) 45,48,48
   45 RETURN
   48 ER=NVB(2)
      RETURN
      END


      SUBROUTINE WHICH ( X,A,ASYMC,CNTFR,GAUSW,IBB,NVB,Z,B,N11  )
C
C       THIS SUBROUTINE IS CALLED IN ORDER TO CHOOSE WHICH OF THE
C     APPROXIMATIONS IS TO BE TRIED FIRST IN CALCULATING GAMMA(A,X)
C
C     THIS SUBROUTINE MAY BE ALTERED - WITHOUT CHANGING GAMMA(     )
C
C     GAUSW,ASYMC,CNTFR ARE SWITCHES WHICH ARE SET EQUAL TO ONE IF THE
C     GAUSSIAN,ASYMPTOTIC, OR CONTINUED FRACTION METHODS ARE TO BE USED.
C
C     IBB  ADDS TO THE UPPER BOUND ON THE COUNT OF THE NUMBER OF
C     ITERATIONS FOR EACH METHOD.    IF IBB=0, BOUND=100.
C
C     THERE NOW FOLLOWS A TABLE OF MEANINGS FOR THE VECTOR NVB.
C
C       NVB(1)  IS THE EXP FOR THE ACCURACY CONTROL FACTOR P
C        P = 2.*10.0**EXP(-7)  UNLESS NVB(1) IS SPECIFIED
C
C
C       NVB(2) FLAGS OVERFLOW.  THE OVERFLOW MAY EFFECT THE
C     RESULTS WHEN NVB(2) IS GREATER THAN THREE.
C
C       NVB(3) COUNTS THE NUMBER OF ITERATIONS OR TERMS ACTUALLY USED
C
C       NVB(4) IS A SWITCH, WHEN NVB(4)=1  INFORMATION ABOUT THE NUMBER
C       OF TERMS NEEDED TO REACH AN ACCURACY OF P. TYPE OF SERIES
C       EXPANSION,VALUES OF GAMMA,ETC.,ARE PRINTED OUT BY THE SUBROUTINE
```

```
C
C        NVB(5) , WHEN NOT EQUAL TO ZERO, CAUSES THE SUBROUTINE TO
C        CALCULATE  GAMMA  SUCESSIVELY BY SEVERAL DIFFERENT METHODS.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION NVB(5)
      IBB=100
      A1=2.D0
      NVB(2)=0
C     NVB(4)=1
      NVB(5)=0
   75 IF(X-2.D2) 80,102,102
   80 IF(A-IDINT(A)) 85,83,85
   83 IF(A-2.D1)99,99,98
   85 IF(A+X-13.D0)102,102,98
   98 IF(X-A) 102,100,100
   99 N11=15.D1/A
  100 ASYMC=1.D0
      GO TO 104
  102  ASYMC=0.D0
  104 IF(X-  A ) 107,105,105
  105 IF(X-1.D0)107,107,106
  106 CNTFR=1.D0
      GO TO 109
  107 CNTFR=0.D0
  109 IF(5.D2 -X ) 111,111,112
  111 TRD=.33333333333333333D0
      A9=9.D0*A
      Z=((X/A)**TRD-1.D0+1.D0/A9-2.D0/(A9*A9))*DSQRT(A9)
      IF(DABS(Z) -A1) 121,121,112
  121 GAUSW=1.D0
      GO TO 113
  112 GAUSW=0.0D0
  113 RETURN
      END

      SUBROUTINE GAUSS(Z,ALPHA)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION AT(6)
      CZZ=200.0D0
      SQR2 = 1.4142135623730950D0
      X=DABS(Z/SQR2)
      AT(1) = .0705230784D0
      AT(2)= .0422820123D0
      AT(3) = .0092705272D0
      AT(4) = .0001520143D0
      AT(5) = .0002765672D0
```

```
      AT(6) =  .0000430638D0
C
      GR=0.D0
      DO 720  J=1,6
      JS=7-J
  720 GR=GR*X  + AT(JS)
      GR =1.D0+ GR*X
      IF(GR  - CZZ) 740,740,730
  730 PH=0.D0
      GO TO 750
  740 PH=1.D0/GR**16
  750 ALPHA =.5D0*PH
      IF(Z) 755,760,760
  755 ALPHA=.5D0+.5D0*(1.D0-PH)
  760 RETURN
      END


      SUBROUTINE GAMMA(A,X,GAM,B,NVB)
C
C         INCOMPLETE GAMMA FUNCTION SUBROUTINE
C
C
C     SUBROUTINE GAMMA(A,X,GAM,B,NVB)  IS A RCUTINE FOR EVALUATING THE
C     NORMALIZED GAMMA FUNCTION
C
C                  GAM(A,X) = GAMMA(A,X)/GAMMA(A)
C
C     FOR ALL REAL X AND A GREATER THAN OR EQUAL TO ZERO.
C
C     ACCURACY IS ABOUT PLUS OR MINUS .000001 EXCEPT FOR LARGE A NEAR X
C     OR A LESS THAN .1 UNLESS OTHERWISE SPECIFIED IN WHICH.
C
C     THE GAMMA INTEGRAL IS FROM X TO INFINITY
C     IT IS ALSO EQUAL TO THE A-1 TH. POISSON SUM
C
C     IN THE PEARSON TABLES OF THE INCOMPLETE GAMMA FUNCTION
C     I(U,P) = 1 - GAM( P+1, U*SQRTF(P+1)  )
C     I( X/(A**(1/2)), A-1 ) = 1 - GAM(A,X)
C
C     AND FOR CHI-SQUARE PERCENTILE TABLES, THE PROBABILITY THAT
C     CHI-SQUARE IS LESS THAN CS     IS
C       PROB( CX, DF ) = 1. - GAM( DF/2 , CX/2 )
C     THAT IS TO SAY X = CX/2  AND A = DF/2
C       OR  CHI-SQUARE = 2*X  AND DF = 2*A
C
C     B  IS THE A-TH. POISSON TERM.
C     IT IS ALSO ACCURATE TO ABOUT .0001 PERCENT.
```

```
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION NVB(5) ,AS(8)
      N11=1
      IF(SWX-1.111DO)110,12C,110
  110 CONTINUE
      AS(1)=-.577191652DO
      AS(2)=0.988205891DO
      AS(3)=-.897056937DO
      AS(4)=0.918206857DO
      AS(5)=-.756704078DO
      AS(6)=C.482199394DO
      AS(7)=-.193527818DO
      AS(8)=0.035868343DO
      E=2.718281828459045DO
      TPI= 6.283185307179586DO
      CZP=--35.0DO
      CZ=1.D1**CZP
      CZR = 1.DO/CZ
      SWX=1.111DO
  120 AFR=A-IDINT(A)
      DEM=A + 1.DO
      F=1.0DO
      XP=-7.DO
      SWS=0.DO
      SWT=0.DO
      SWW=0.DO
      SWR=0.DO
      SFS=0.DO
      SWG=0.DO
      Z=0.DO
      NVB(2)=0
      NS=0
      NT=0
      NR=0
      BT= .4342944819 DO
      X1 = (DSQRT(A))*(11.DO+5.DO*(BT*DLOG(A)-.9DO)**2 )
  148 IF(X-CZ) 149,149,150
  149 B=0.DO
      GAM=1.DO
      GAMT = 1.DO
      SWT=1.DO
      GO TO 400
C
  150 IF(A-0.5DO) 155,155,151
  151 IF(A - 4.D1) 152,152,190
  152 IF(X - X1) 156,156,153
C
```

```
  153 B=0.DO
  154 GAM=0.DO
      SWW=1.0DO
      GO TO 400
C
  155 IF(X - 85.DO)156,156,153
C
C
C     CALCULATION OF THE POISSON TERM B(X,A)
  156 DO 170 I=1,50
      DEM=DEM-1.0DO
      IF(DEM-1.0DO) 175,159,160
C
  159 F=X*F
      GO TO 175
  160 F=(X/DEM)*F
  163 IF(F-CZ) 153,170,170
  170 CONTINUE
  171 WRITE (6,901)
      RETURN
C
  175 EXPX = E**X
      GGF=DLOG(F/EXPX)
C
C     GAMMA FUNCTION OF ONE PLUS THE FRACTIONAL PART OF A
C     FROM CHEBYSCHEV APPROXIMATION OF POWER 8 (HASTINGS)
C     ACCURATE TO WITHIN + OR - .0000 002
C
  176 GS=0.0DO
      DO 180 J=1,8
      JS=9-J
  180 GS=GS*AFR+AS(JS)
      GA=1.DO+GS*AFR
C
  181 GGB= AFR*DLOG(X)  + GGF - DLOG(GA)
      GO TO 193
C
  187 B=0.DO
      BAM=0.DO
      GO TO 196
C
C     STIRLINGS APPROXIMATION EXTENDED
  190 WPT=DLOG(X) - DLOG(A) +1.DO- X/A
      GGB=A*WPT-.5DO*(DLOG(TPI)+DLOG(A))-1.DO/(12.DO*A)
     1+1.DO/(360.DO*A**3)
  193 IF (GGP+85.DO) 187,194,194
  194 B= E**GGB
  195 BAM=(A/X)*B
```

```
196 CALL    WHICH ( X,A,ASYMC,CNTFR,GAUSW,IBB,NVB,Z,B,N11 )
    V=NVB(1)
    IF(NVB(1)) 125,130,125
125 XP=-DABS( V)
130 P=2.D0*10.D0**XP
    IUB=IBB + 100
197 IF(GAUSW) 198,198,700
198 IF(ASYMC) 199,199,300
199 IF(CNTFR) 200,200,600
C
C
C     CONVERGENT SERIES FOR SMALL X
200 SWT=1.D0
201 T=1.D0
202 SUMT=1.D0
203 DO 230 NT=1,IUB
204 ZT=NT
205 T=T*X/(A + ZT )
206 RAT = T/SUMT
207 IF(RAT-P) 240,240,220
220 SUMT=SUMT+T
221 IF(SUMT - CZR) 230,230,235
230 CONTINUE
    WRITE            (6, 904)
    NVB(2) = 5
    GO TO 240
C   PRINT 903
235 NVB(2)=6
240 GAMT=1.D0-B*SUMT
    GAM=GAMT
    GO TO 400
C
C     ASYMPTOTIC SERIES FOR LARGE X, A GREATER THAN 1
300 SWS=1.D0
    S=1.D0
    SUMS=1.D0
    XM=-X
C
315 DO 330 NS=1,IUB
316 ZS=NS
317 CE=(ZS-A)/XM
348 NACE = DABS(CE)
319 IF(NACE-N11 ) 320,340,340
320 S=CE*S
321 RAS=DABS(S/SUMS)
322 IF(RAS-P) 350,350,330
330 SUMS=SUMS+S
    WRITE            (6,904)
```

```
C
      NVB(2) = 4
  340 GAMS=BAM*SUMS
      GO TO 199
C
  350 GAMS=BAM*SUMS
      GAM=GAMS
      IF(NVB(5) -2) 400,400,199
C
C     CONTINUED FRACTION APPROX. TO GAMMA FOR X LARGE RELATIVE TO A
C       ODD PART
  600 SWR = 1.DO
  601 AAM=1.DO/X
  602 BBM=1.DO
  603 AA= (X+1.DO)/(X*X)
  604 BB=(X+2.DO-A)/X
  605 APPX=AA/BB
  620 DO 647 N=2,IUB
  621 SN=N
  622 AC= -(SN-1.DO)*(SN-A)/(X*X)
  623 BC= (X+2.DO*SN-A)/X
  624 AAP=BC*AA+AC*AAM
  625 AAA=DABS(AAP)
  642 IF(AA-CZR) 643,643,655
  643 CONTINUE
  644 BBP=BC*BB+AC*BBM
  645 BBB=DABS(BBP)
  632 IF(BBB-CZR) 633,633,655
  633 CONTINUE
  634 AAM=AA
  635 AA=AAP
  636 BBM=BB
  637 BB=BBP
  638 APPXM=APPX
  639 APPX=AA/BB
      RAR=DABS((APPX-APPXM)/APPXM)
  641 IF(RAR-P) 650,650,647
  647 CONTINUE
  650 NR=3*N
  651 CFCT=A*APPX
      GO TO 660
  655 GAMR=B*A*APPX
C
  661 SFS=1.DO
      NVB(2)=3
      NR=3*N
  662 CNTFR=0.DO
      GO TO 300
```

```
  66C GAMR=B*CFCT
      IF(CFCT) 661,661,665
  665 GAM=GAMR
      IF(NVB(5)-3) 400,400,662
C
C
C
  700 SWG=1.DO
      IF(Z) 710,705,710
  705 TRD=.3333333333333333DO
      A9=9.DO*A
      Z=((X/A)**TRD-1.DO+1.DO/A9-2.DO/(A9*A9) )*DSQRT(A9)
  710 CALL GAUSS(Z,GAAM)
      IF(GAAM) 775,780,780
  775 NVB(2)=2
      GO TO 198
  780 IF(NVB(5))785,785,198
  785 GAM=GAAM
      GO TO 400
C
C
  400 IF(NVB(4)-1)401,402,401
  401 GO TO 900
  402 IF(SWZ-1.1111DO) 403,405,403
  403 SWZ=1.1111DO
  404 WRITE              (6,911)P
  405 WRITE              (6,906)
      IF(SWS-1.DO)420,418,420
  418 WRITE              (6,918)NVB(2),NS,    RAS,X1,GAMS,GA,B,X,A
  420 IF(SWR-1.DO)433,431,433
  431 IF(SFS-1.DO)4316,4315,4316
 4315 WRITE              (6,962)RAR,NR
 4316 WRITE              (6,948)NVB(2),NR,    RAR,X1,GAMR,GA,B,X,A
  433 IF(SWT-1.DO) 430,428,430
  428 WRITE              (6,928)NVB(2),NT,     RAT,X1,GAMT,GA,B,X,A
  430 IF(SWW-1.DO) 440,438,440
  438 WRITE              (6,938)X1,GAM,X,A
  440 IF(SWG-1.DO) 450,448,450
  448 RAG=(GAAM - GAM)/GAM
C     WRITE OUTPUT TAPE 6,958,Z,RAG,X1,GAAM,GA,B,X,A,
      WRITE              (6,958)Z,RAG,X1,GAAM,GA,B,X,A
  450 GO TO 900
C
  900 NVB(3) = NS + NT + NR
      RETURN
C
  901 FORMAT(20H LOOP IN CALC. OF B                                      )
  902 FORMAT(55H  FLOATING POINT UNDER-SPILL GAMMA                        )
```

```
903 FORMAT(50H POSSIBILITY OF FLOATING POINT OVERFLOW IN GAMMA      )GAMA3880
962 FORMAT(84H FLOATING POINT SPILL IN CONTINUED FRACTION CALCULATION GAMA3890
   1 OF GAMMA.  RELATIVE ERROR  F10.7,5X,I3,17H TERMS CALCULATED    )GAMA3900
904 FORMAT(50H UPPER BOUND ON ITERATION COUNT REACHED IN GAMMA      )GAMA3910
906 FORMAT(1H  )                                                     GAMA3920
911 FORMAT(120H  TYPE OF                 NUMBER      Z      LAST TERM  GAMA3930
   1X1      GAMMA(X,A)    GAMMA (UNE +      B            X          A /GAMA3940
   2        115H EXPANSION   NVB(2)  OF TERMS            OVER   SUM    GAMA3950
   3        /GAMMA(A)     FRCT.PART A)                                /GAMA3960
   4   38X, F9.7              )                                        GAMA3970
928 FORMAT( 12H CONVERGENT, I5, 4X, I4, 3X, 9X  ,    F10.7, 3X,        GAMA3980
   1  F8.4 , 3X, F10.7, 3X, F10.7,1X,F13.7, 3X, F7.3, 2X, F8.4  )      GAMA3990
918 FORMAT( 12H ASYMPTOTIC, I5, 4X, I4, 3X, 9X  ,    F10.7, 3X,        GAMA4000
   1  F8.4 , 3X, F10.7, 3X, F10.7,1X,F13.7, 3X, F7.3, 2X, F8.4  )      GAMA4010
938 FORMAT (14H X EXCEEDS X1 36X, F8.4, 3X, F10.7,31X,F7.3, 2X,F8.4 )GAMA4020
948 FORMAT( 12H CONT.FRACT. I5, 4X, I4, 3X, 9X  ,    F10.7, 3X,        GAMA4030
   1  F8.4 , 2X, F11.7, 3X, F10.7,1X,F13.7, 3X, F7.3, 2X, F8.4  )      GAMA4040
958 FORMAT(12H GAUSSIAN     5X,6X,F11.7,3X,F10.7, 3X,                  GAMA4050
   1  F8.4 , 2X, F11.7, 3X, F10.7,1X,F13.7, 3X, F7.3, 2X, F8.4  )      GAMA4060
    END                                                                GAMA4070
```

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UNCLASSIFIED

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DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Naval Postgraduate School
Monterey, California 93940

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

3. REPORT TITLE

A COMPARISON OF THE EXACT AND APPROXIMATE POWER OF THE CHI-SQUARE GOODNESS-OF-FIT TEST

4. DESCRIPTIVE NOTES (Type of report and, inclusive dates)

Master's Thesis; March 1971

5. AUTHOR(S) (First name, middle initial, last name)

Brian Theodore Wright, Captain, United States Marine Corps

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 1971	70	12

8a. CONTRACT OR GRANT NO.

b. PROJECT NO.

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Naval Postgraduate School
Monterey, California 93940

13. ABSTRACT

This thesis presents a numerical comparison of the exact and approximate powers of the chi-square goodness-of-fit test for small numbers of classes and small sample sizes for the equiprobable null hypothesis. The comparison was performed using an IBM 360 computer and the computational details are presented within the thesis. In addition a comparison of critical points was conducted for the chi-square distribution and the associated exact, (multinomial), distribution. The results of the power comparisons show that the approximate power is surprisingly good and is recommended as an efficient method for determining type two error associated with the test. Further, use of the chi-square distribution for determining a critical point is reinforced through the numerical comparison of significance levels.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

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A-31408

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
CHI-SQUARE						
GOODNESS-OF-FIT TEST						
POWER						

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